Partially Layer-Wise Advanced Zig-Zag and HSDT Models Based on the Generalized Unified Formulation

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The Generalized Unified Formulation (GUF) for composites and sandwich structures is a computational multi-theory architecture which can generate a large variety of types of theories with any combination of the orders of expansion for the different primary variables. All the theories are generated from the expansion of six theory-invariant $1 \times 1$ arrays (kernels). For the first time, an extension and further generalization of the GUF is proposed: each displacement unknown is independently described (in an axiomatic sense) with respect the other primary variables. In particular, each displacement can present a Layer Wise or an Equivalent Single Layer types of discretizations with or without the Zig-Zag enhancement which takes into account the displacements’ thickness derivative interlaminar discontinuity.

The Partially Zig-Zag Advanced Higher Order Shear Deformation Theories (PZZAHSDTs), Partially Layer-Wise Higher Order Shear Deformation Theories (PLHSDTs), Partially Layer-Wise Advanced Higher Order Shear Deformation Theories (PLAHSHTs), Partially Layer-Wise Advanced Zig-Zag and Higher Order Shear Deformation Theories (PLAZZHS-DTs) are introduced and discussed for the first time. These types of theories are compared with the advanced Equivalent Single Layer, Zig-Zag, and Layer Wise approaches. In the framework of displacement-based models, a Finite Element implementation of the Generalized Unified Formulation is presented for the first time and numerical evaluations of sandwich structures subjected to various boundary conditions are discussed. A challenging high Face-to-Core Stiffness Ratio (FCSR) is also adopted to assess the performances of this large set of new approaches. It is demonstrated that the Partially-Layerwise Theories provide a computationally efficient alternative to standard Layer-Wise models.

I. Introduction

COMPOSITES generally present significant changes of material properties in the thickness direction. Anisotropy and shear deformation effects need to be taken into account. The Classical Plate Theory (CPT)\(^1\) is an effective model for metallic thin panels. However, for moderately thick structures it is inadequate. This led to the development of alternative approaches in which the CPT assumptions were relaxed. In particular, the First Order Shear Deformation Theory (FSDT)\(^2–4\) was introduced. The generic planar cross section initially orthogonal to the mid-plane of the plate is assumed to be planar after the deformation takes place. However, it is no longer perpendicular to the mid-surface of the plate. FSDT is a good compromise but is not sufficient for more challenging cases with localized effects. Therefore, it was proposed to remove the assumption of planar cross section in the deformed state. In other words, an initially planar cross section is allowed to deform in a generic shape. This is accomplished with an axiomatic description (usually for the in-plane displacements only) which includes higher order terms in the thickness expansion. These types of theories are called Higher Order Shear Deformation Theories (HSDT).\(^5–11\) However, to correctly represent the three-dimensional displacement and stress fields it is necessary to include the transverse strain effects. This can be accomplished by including higher order terms in the thickness expansion for the out-of-plane displacement $u_z$. This type of theories is often indicated as Advanced Higher Order Shear Deformation Theories (AHSDT). However, due to both the interlaminar equilibrium of the transverse stresses and anisotropy of the mechanical properties along the thickness, all the displacement and stress variables (with the only exception of the transverse normal stress) present a discontinuity of the first derivative with respect to the thickness coordinate $z$. Especially for the displacement variables this Zig-Zag functional form is defined

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as “Zig-Zag form of the displacements.” Many researchers proposed to include the Zig-Zag form of the displacements a priori with the goal of having a computationally inexpensive but more efficient formulation. Following the historical reconstruction attempted in Reference [15] on this subject, the Zig-Zag theories can be subdivided into 3 major groups:

- Lekhnitskii Multilayered Theory (LMT)
- Ambartsumian Multilayered Theory (AMT)
- Reissner Multilayered Theory (RMT)

In the Lekhnitskii Multilayered Theory (originally formulated for multilayered beams) the Zig-Zag form of the displacements and continuity of the transverse stresses were enforced. LMT was extended to the case of plates by Ren. In the Ambartsumian Multilayered Theory an interlaminar continuous transverse shear stress field is a priori enforced. The displacement fields present a discontinuity of the first derivatives in the thickness direction. Later the effects of transverse normal strain/stress were also included. Whitney applied AMT to anisotropic and non-symmetrical plates. Later Rath and Das extended Whitney’s work to shells and dynamic problems.

In the Reissner Multilayered Theory the transverse stresses are primary unknowns as well as the displacement variables. The variational statement is Reissner’s Mixed Variational Theorem. Murakami proposed to take into account the Zig-Zag effects by enhancing the corresponding displacement variable with a Zig-Zag function denoted here as Murakami’s Zig-Zag Function (MZZF). Numerous applications of the concept of enhancing the displacement field with MZZF have been presented. This enhancement provides a significant improvement of the accuracy with little increment of the computational cost with respect to the inexpensive (but less accurate) classical methods.

Recently researchers adopted Zig-Zag models to solve various problems involving bending analysis of functionally graded sandwich structures, laminated beams, and buckling problems. The effectiveness of taking into account the displacements’ slope discontinuity at the interfaces with Zig-Zag models has been proven. For a detailed quasi-3D type of investigation a Layer Wise approach represents a valuable alternative to the computationally demanding Finite Element approaches based on solid elements. The Layer Wise and Equivalent Single Layer approaches can be unified with the adoption of the Compact Notations (CN). Examples of Compact Notations are represented by Carrera’s Unified Formulation (CUF) and its generalization represented by the Generalized Unified Formulation (GUF). In the GUF each displacement variable (or/and stress variable in the case of mixed formulations) are independently represented and any combination of orders can be achieved. For example, an AHSDT with a fourth order thickness expansion for the in-plane displacements and a parabolic expansion for the out-of-plane displacement can be represented as well as a LW theory with cubic expansion of the in-plane displacement variables and parabolic expansion for the transverse displacement $u_z$.

Up to now the Generalized Unified Formulation could handle any combination of orders for the displacement unknowns. The type of description was the same for all the displacements. With this work GUF is further generalized as follows:

- One or more primary variables is enhanced with Murakami’s Zig-Zag Function. The other variables present a HSDT-type description.
- One or more primary variables is described in a Layerwise sense. The other variables present a HSDT-type description.
- Layerwise, HSDT-type description, and Zig-Zag enhancements are all present at the same time.

In all these cases any order of expansion is possible. For example, with this new generalization, the Generalized Unified Formulation could include a theory which presents a cubic Equivalent Single Layer expansion for the displacement $u_z$, a parabolic layerwise description for the displacement $u_y$ and a linear Zig-Zag enhanced Equivalent Single Layer description of the variable $u_z$.

With the extension presented in this work GUF is more general and could have practical applications in optimization and probabilistic studies or in the quasi-3D investigations and analysis of local effects. This means that the user has an extremely versatile multi-theory and multi-fidelity architecture which can generate
practically any axiomatic theory according to the desired level of accuracy and computational time required by the problem under investigation.

This paper first presents a classification of the theories with the inclusion of new types of intermediate approaches. Then the finite element GUF representation of these formulations are presented with particular emphasis on the assembling in the thickness direction. Finally, several numerical examples with different boundary conditions are examined and compared with the literature and elasticity solutions. The test cases are investigated with classical, high fidelity and intermediate-fidelity models and several indications and guidelines on how to select the orders and type of discretizations are provided.

II. Axiomatic Formulation for Composites and Sandwich Structures

An accurate three-dimensional evaluation of the displacement and stress fields (including the transverse shear and normal stresses) of composites and sandwich structures is crucial for a reliable prediction of the structural performance. Thus, researchers formulated a large amount of axiomatic theories: starting from the main physical observations of the structural behavior, the equations are simplified. The simplification, if properly accomplished, allows one to capture the main aspects with an optimal use of the computational resources.

A successful axiomatic model is the one that takes into account the major physical aspects and neglects what is not determinant for a given problem (geometry, boundary conditions and loadings are assigned). For a multilayered composite structure these aspects could be summarized as follows.

- All the displacements $u_x$, $u_y$, and $u_z$ must be continuous functions in the thickness direction (compatibility conditions).
- All the transverse stresses $\sigma_{zx}$, $\sigma_{zy}$, and $\sigma_{zz}$ must be continuous functions in the thickness direction (equilibrium conditions).
- All the in-plane stresses $\sigma_{xx}$, $\sigma_{xy}$, and $\sigma_{yy}$ are in general discontinuous functions at the interface between layers.
- All the in-plane strains must be continuous at the interface between layers. This can be shown to be a consequence of the compatibility of the displacement fields.
- The transverse strains may be discontinuous functions at the interface between layers $k$ and $k + 1$ without violating the compatibility of the displacements (see and compare Figures 1 and 2 which present the in-plane and transverse shear strains).

Figure 1. Interlaminar discontinuity of the in-plane shear strain and violation of the compatibility of the displacement field.

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Figure 2. Interlaminar discontinuity of the transverse shear strain and satisfied compatibility of the displacement field.

- All the displacements $u_x$, $u_y$, and $u_z$ present a Zig-Zag form (first thickness derivative of the displacements is in general not a continuous function at the interfaces between 2 consecutive layers). The Zig-Zag form of the in-plane displacements can be directly demonstrated from the equilibrium conditions of the transverse shear stresses. The Zig-Zag form of the transverse displacement $u_z$ can be directly derived from the equilibrium condition of the transverse normal stress $\sigma_{zz}$.

- The transverse shear stresses $\sigma_{xz}$ and $\sigma_{zy}$ present a Zig-Zag form (first thickness derivative of these stresses is not a continuous function at the interfaces between adjacent layers). This can be shown starting from the first 2 equilibrium equations and Hooke’s equation written in its mixed form.

- The transverse normal stress $\sigma_{zz}$ does not present any Zig-Zag pattern along the thickness even if the materials and/or fiber orientations adopted for the different layers are different. This property can be easily deduced from the third equilibrium equation and can be shown to be a direct consequence of the equilibrium condition for the transverse shear stresses $\sigma_{xz}$ and $\sigma_{zy}$.

- For “thin” plates (small thickness compared to the plate dimensions) the Zig-Zag form of the displacements is not very significant.

It is clear that a theory which satisfies all the above discussed properties will probably present a very good prediction of the displacement, stress, and strain fields. Another theory obtained from the previous one with the removal of some of the above discussed features will not be able to predict the structural behavior with the same degree of accuracy. However, it is probably computationally inexpensive and could be used in situations when the discarded properties are less important. For example, if a plate is very thin it is probably not necessary to have a theory able to predict the Zig-Zag form of the displacements since by direct observation of the typical deformation of thin plates this behavior could be safely discarded. Moreover, from direct observation it could be observed that the bending of the very thin plate is usually characterized by a preservation of the right angle between the normal directions and the mid-plane of the plate even in the deformed configuration. This is the main observation that leads to the so called Classical Plate Theory.
A. Classical Plate Theory

The displacement field is a priori prescribed as follows:

\[
\begin{align*}
    u_x(x, y, z) &= u_{x0}(x, y) - z \frac{\partial u_{z0}(x, y)}{\partial x} \\
    u_y(x, y, z) &= u_{y0}(x, y) - z \frac{\partial u_{z0}(x, y)}{\partial y} \\
    u_z(x, y, z) &= u_{z0}(x, y)
\end{align*}
\]

(1)

It should be noted that the unknowns of the problem are the functions \(u_{x0}(x, y), u_{y0}(x, y),\) and \(u_{z0}(x, y):\) with their knowledge the displacements at any point identified by the coordinates \(x, y,\) and \(z\) can be calculated. It should also be observed that the number of unknowns does not change if the number of layers is changed. For example, in a three-layered sandwich structure the number of unknowns is equal to three.

B. First Order Shear Deformation Theory

When the plate is relatively thick the three-dimensional effects are more important and the displacement field of equation 1 is inadequate. For example, the cross section may still be considered planar but is no longer perpendicular to the mid-surface in the deformed state. In other words, the plate experiences transverse shear strains: \(\gamma_{zx} \neq 0, \gamma_{zy} \neq 0.\) A possible model reflecting this fact could be given by the following displacement field:

\[
\begin{align*}
    u_x(x, y, z) &= u_{x0}(x, y) + z\phi_{u_z}(x, y) \\
    u_y(x, y, z) &= u_{y0}(x, y) + z\phi_{u_y}(x, y) \\
    u_z(x, y, z) &= u_{z0}(x, y)
\end{align*}
\]

(2)

It should be noted that the unknowns of the problem are the five functions \(u_{x0}(x, y), u_{y0}(x, y), u_{z0}(x, y), \phi_{u_z}(x, y),\) and \(\phi_{u_y}(x, y):\) with their knowledge the displacements at any point identified by the coordinates \(x, y,\) and \(z\) can be calculated.

C. First Order Shear Deformation Theory with Transverse Strain Effects Included: Advanced First Order Shear Deformation Theory (AFSDT)

The transverse strain effects could be retained with a modification of equation 2 as follows:

\[
\begin{align*}
    u_x(x, y, z) &= u_{x0}(x, y) + z\phi_{u_z}(x, y) \\
    u_y(x, y, z) &= u_{y0}(x, y) + z\phi_{u_y}(x, y) \\
    u_z(x, y, z) &= u_{z0}(x, y) + z\phi_{u_z}(x, y)
\end{align*}
\]

(3)

It is immediate to verify that the transverse normal strain \(\epsilon_{zz}\) is no longer zero:

\[
\epsilon_{zz} = \frac{\partial u_z}{\partial z} = \frac{\partial}{\partial z} [u_{z0}(x, y) + z\phi_{u_z}(x, y)] = \phi_{u_z}
\]

(4)

In equation 3 the number of unknowns is equal to six regardless of the number of layers. In particular, the unknowns are the following: \(u_{x0}(x, y), u_{y0}(x, y), u_{z0}(x, y), \phi_{u_z}(x, y), \phi_{u_y}(x, y),\) and \(\phi_{u_z}(x, y).\)

An Advanced First Order Shear Deformation Theory is a First Order Shear Deformation Theory in which the transverse normal strain \(\epsilon_{zz}\) is not zero.

D. Higher-Order Shear Deformation Theories (HSDTs)

In more complex situations CPT and FSDT are no longer adequate and an initially normal straight line does not remain a straight line after the deformation. This means that higher-order terms are needed in the
seven of the in-plane displacements $u_z$ place. This is the reason why higher order powers of $z$ are used to clearly indicate that the normal straight line is no longer straight after the deformation takes place. This is the reason why higher order powers of $z$ (thickness coordinate) in the axiomatic expansion of the in-plane displacements $u_x$ and $u_y$ are adopted. In equation 5 the number of unknowns is equal to seven. In particular, the unknowns are the following: $u_{x0}(x,y)$, $u_{y0}(x,y)$, $u_{z0}(x,y)$, $\phi_{1_{uz}}(x,y)$, $\phi_{1_{uy}}(x,y)$, $\phi_{2_{uz}}(x,y)$, and $\phi_{2_{uy}}(x,y)$. The following definition will be used:

A Higher Order Shear Deformation Theory presents non-zero and non-constant transverse shear strains in the thickness direction. The transverse normal strain $\epsilon_{zz}$ is equal to zero everywhere.

E. Advanced Higher-Order Shear Deformation Theories (AHSDTs)

It should be noted that other Higher Order Shear Deformation Theories could be formulated. For example this is a possible option:

$$
\begin{align*}
    u_x(x,y,z) &= u_{x0}(x,y) + z\phi_{1_{uz}}(x,y) + z^2\phi_{2_{uz}}(x,y) + z^3\phi_{3_{uz}}(x,y) \\
    u_y(x,y,z) &= u_{y0}(x,y) + z\phi_{1_{uy}}(x,y) + z^2\phi_{2_{uy}}(x,y) + z^3\phi_{3_{uy}}(x,y) \\
    u_z(x,y,z) &= u_{z0}(x,y) + z\phi_{1_{uz}}(x,y) + z^2\phi_{2_{uz}}(x,y)
\end{align*}
$$

In equation 6 the number of unknowns is equal to eleven. The following definition will be used:

An Advanced Higher Order Shear Deformation Theory presents non-zero and non-constant transverse shear strains in the thickness direction. The transverse normal strain $\epsilon_{zz}$ is also different than zero.

F. Zig-Zag Theories (ZZTs)

It has been discussed that the different mechanical properties of two adjacent layers implies the discontinuity of the first derivative of the displacements (zig-Zag form of the displacements). This concept is explained in Figure 3 for a three-layered structure. All the axiomatic models presented in equations 1, 2, 3, 5, and 6 violate this physical property which is increasingly important if the thickness dimension of the plate is increased. The explicit demonstration of this fact is now presented for the theory reported in equation 5. Consider the axiomatic expansion adopted for the displacement $u_x$:

$$
\begin{align*}
    u_x(x,y,z) &= u_{x0}(x,y) + z\phi_{1_{ux}}(x,y) + z^2\phi_{2_{ux}}(x,y)
\end{align*}
$$

The derivative of the displacement $u_x$ evaluated at the top surface of layer $k$ can be expressed from equation 7:

$$
\left[ \frac{\partial u_x}{\partial z} \right]_{z=z_{topk}} = \frac{\partial u_x^{k,t}}{\partial z} = \phi_{1_{ux}}(x,y) + 2z_{topk}\phi_{2_{ux}}(x,y)
$$

From equation 7 the slope of the same displacement evaluated at the bottom surface of layer $k + 1$ can also be obtained:

$$
\left[ \frac{\partial u_x}{\partial z} \right]_{z=z_{bot(k+1)}} = \frac{\partial u_x^{k+1,b}}{\partial z} = \phi_{1_{ux}}(x,y) + 2z_{bot(k+1)}\phi_{2_{ux}}(x,y)
$$

However, by definition of interface the $z$ coordinate at the interface has a unique value: $z_{bot(k+1)} = z_{topk}$. This implies that equation 9 can be rewritten as

$$
\frac{\partial u_x^{k+1,b}}{\partial z} = \phi_{1_{ux}}(x,y) + 2z_{topk}\phi_{2_{ux}}(x,y)
$$

\[10\]

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which presents the same right hand side of equation 8. Thus, it is deduced that equation 7 which reports a typical expansion for $u_x$ implies no discontinuity of the slope of displacement $u_x$ in the thickness direction (no Zig-Zag form is then present and this contradicts a physical requirement; see also Figure 3). Similar arguments could be adopted for the other theories previously discussed.

How is it possible to introduce in an axiomatic sense and a priori the discontinuity of the first derivative at the interface between layers? One effective option is to increase the number of unknown functions used to describe the displacements by adding some terms which enforce the discontinuity of the slope. This is demonstrated here for the example reported in equation 7 which presents a parabolic expansion for $u_x$. It should be noted that in equation 7 the number of unknown functions is equal to three. They are the following: $u_{x0}(x, y)$, $\phi_{1u_x}(x, y)$, and $\phi_{2u_x}(x, y)$. Equation 7 is now modified by employing an additional unknown function $u_{xz}(x, y)$:

$$u_x(x, y, z) = u_{x0}(x, y) + z\phi_{1u_x}(x, y) + z^2\phi_{2u_x}(x, y) + (−1)^k \zeta_{uk} u_{xz}(x, y) \tag{11}$$

The coefficient $−1^k \zeta_k$ which multiplies the new unknown function $u_{xz}(x, y)$ enforces the a priori discontinuity of the first derivative. To show that it is necessary to write its explicit form. First, it should be noted that $k$ is the identity of the layer under investigation. For example, in a three-layered structure the bottom layer will have $k = 1$, the mid-layer will have $k = 2$, and the top layer will have $k = 3$. The term $−1^k$ is then equal to $−1$ and +1 (alternatively). $\zeta_k$ is a dimensionless thickness coordinate defined at layer level. It is related with the physical $z$ coordinate (measured from the mid-plane of the plate) by the following relation:

$$\zeta_k = \frac{2}{z_{top_k} - z_{bot_k}} z - \frac{z_{top_k} + z_{bot_k}}{z_{top_k} - z_{bot_k}} \tag{12}$$

where $z_{bot_k}$ indicates the thickness coordinate of the bottom surface of layer $k$ and $z_{top_k}$ indicates the thickness coordinate of the top surface of layer $k$. Note that $−1 \leq \zeta_k \leq +1$. To show that equation 11 implies an a-priori enforced discontinuity of the first derivative, it is useful to calculate the derivative at the top surface of layer $k$ and bottom surface of layer $k + 1$. It is possible to write:

$$\frac{\partial u_{z(k+1)b}}{\partial z} = \phi_{1u_x} + 2z_{bot_{(k+1)}}\phi_{2u_x} + (−1)^{(k+1)} \frac{\partial \zeta_{(k+1)}}{\partial z} u_{xz}$$

$$\frac{\partial u_{zt}}{\partial z} = \phi_{1u_x} + 2z_{top_k}\phi_{2u_x} + (−1)^k \frac{\partial \zeta_k}{\partial z} u_{xz} \tag{13}$$
or (see equation 12 for the definition of \( \zeta_k \))

\[
\begin{align*}
\frac{\partial u_x^{(k+1) b}}{\partial z} &= \phi_{1_{ux}} + 2z_{bot(k+1)}\phi_{2_{ux}} + (-1)^{(k+1)} \frac{2}{z_{top(k+1)} - z_{bot(k+1)}} u_{xz} \\
\frac{\partial u_x^{k t}}{\partial z} &= \phi_{1_{ux}} + 2z_{top_k}\phi_{2_{ux}} + (-1)^k \frac{2}{z_{top_k} - z_{bot_k}} u_{xz}
\end{align*}
\]

These relations can be further elaborated by introducing \( h_{(k+1)} = z_{top(k+1)} - z_{bot(k+1)} \) (thickness of layer \((k + 1)\)) and \( h_k = z_{top_k} - z_{bot_k} \) (thickness of layer \(k\)) and using the fact that \( z_{bot(k+1)} = z_{top_k} \) (this is true by definition of interface):

\[
\begin{align*}
\frac{\partial u_x^{(k+1) b}}{\partial z} &= \phi_{1_{ux}} + 2z_{top_k}\phi_{2_{ux}} + (-1)^{(k+1)} \frac{2}{h_{(k+1)}} u_{xz} \\
\frac{\partial u_x^{k t}}{\partial z} &= \phi_{1_{ux}} + 2z_{top_k}\phi_{2_{ux}} + (-1)^k \frac{2}{h_k} u_{xz}
\end{align*}
\]

since the exponent \( k \) is an integer it is always \((-1)^{(k+1)} = -(-1)^k\) thus, it is possible to deduce from equation 15:

\[
\begin{align*}
\frac{\partial u_x^{(k+1) b}}{\partial z} = \frac{\partial u_x^{k t}}{\partial z} - 2 (-1)^k u_{xz} \left[ \frac{1}{h_k} + \frac{1}{h_{(k+1)}} \right] \Rightarrow \frac{\partial u_x^{(k+1) b}}{\partial z} \neq \frac{\partial u_x^{k t}}{\partial z}
\end{align*}
\]

The following should be observed:

- The axiomatic description of \( u_x \) reported in equation 7 does not present the discontinuity (at the interface between two generic layers) of the first derivative with respect to \( z \).
- The inclusion of an additional DOF [and in particular the term \((-1)^k \zeta_k u_{xz} (x, y)\)] enforces a priori the discontinuity of the first thickness derivative (see equation 16). Thus, the model is more suitable for the description of relatively thick plates in which the Zig-Zag form of the displacements is more important.
- Similar formulation could be repeated for the other displacement variables \( u_y \) and \( u_z \) and enforce their Zig-Zag form with the additional unknowns.

The additional term \((-1)^k \zeta_k u_{xz} (x, y)\) is called Murakami’s Zig-Zag Function (MZZF). As an example the fully Zig-Zag counterpart of equation 5 is the following displacement field:

\[
\begin{align*}
&u_x (x, y, z) = u_{x_0} (x, y) + z\phi_{1_{ux}} (x, y) + z^2\phi_{2_{ux}} (x, y) + (-1)^k \zeta_k u_{xz} (x, y) \\
&u_y (x, y, z) = u_{y_0} (x, y) + z\phi_{1_{uy}} (x, y) + z^2\phi_{2_{uy}} (x, y) + (-1)^k \zeta_k u_{yz} (x, y) \\
&u_z (x, y, z) = u_{z_0} (x, y) + (-1)^k \zeta_k u_{zz} (x, y)
\end{align*}
\]

In equation 17 the number of unknowns is now \( 7 + 3 = 10 \).

G. Equivalent Single Layer (ESL) Theories

In the Classical Plate Theory, First Order Shear Deformation Theory, Higher Order Shear Deformation Theories and Zig-Zag Theories a single displacement field is axiomatically described in the thickness direction. In particular, the description of the displacement variables in the thickness direction \( z \) is an axiomatic expansion (in the previous examples the chosen function was a Taylor polynomial) of a selected degree (constant, linear, parabolic, cubic etc.). It is important to realize that the Zig-Zag theories have a single displacement field in the thickness direction even if the coefficient \((-1)^k \zeta_k\) of Murakami’s Zig Zag Function changes when a different layer is considered.

All theories in which a single displacement field is axiomatically prescribed for the multilayered plate, are called Equivalent Single Layer (ESL) Theories. The following definition is adopted:

A theory in which all its unknown functions do not depend on the considered layer is called Equivalent Single
I. Advanced Zig-Zag Theories (AZZTs)

a zero transverse normal strain \( \epsilon \)

Since all the unknown displacement functions are layer-dependent, a superscript \( A \) theory in which all its unknown functions do depend on the considered layer is called Layer-Wise theory (LW). This is the case of the so-called Layer-Wise (LW) theories. The following definition is introduced:

For a quasi-three-dimensional accuracy of the solution the ESL theories are often not sufficiently accurate. This is especially true if the stresses need to be computed and the plate has general geometry, boundary conditions, and loads (maybe localized in a small portion of the panel). For that reason, an axiomatic displacement field can be prescribed for each layer (not for the entire plate at multilayered level as done in ESL case). This is the case of the so-called Layer-Wise (LW) theories. The following definition is introduced:

A Zig-Zag Theory (ZZT) is obtained from an Equivalent Single Layer Theory (called baseline theory) by enhancing all displacements \( u_x, u_y, u_z \) with Murakami’s Zig-Zag Function. The baseline theory presents a zero transverse normal strain \( \epsilon_{zz} \). ZZT is also an Equivalent Single Layer Theory.

I. Advanced Zig-Zag Theories (AZZTs)

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J. Layer-Wise (LW) Theories

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A theory in which all its unknown functions do depend on the considered layer is called Layer-Wise theory. Since all the unknown displacement functions are layer-dependent, a superscript \( k \) will be used to identify
the displacement field axiomatically described at layer level:

\[
\begin{align*}
  u^k_x (x, y, z) &= f^k_{u_{1x}} (z) u^k_{x1} (x, y) + f^k_{u_{2x}} (z) u^k_{x2} (x, y) + f^k_{u_{3x}} (z) u^k_{x3} (x, y) + \ldots \\
  u^k_y (x, y, z) &= f^k_{u_{1y}} (z) u^k_{y1} (x, y) + f^k_{u_{2y}} (z) u^k_{y2} (x, y) + f^k_{u_{3y}} (z) u^k_{y3} (x, y) + \ldots \\
  u^k_z (x, y, z) &= f^k_{u_{1z}} (z) u^k_{z1} (x, y) + f^k_{u_{2z}} (z) u^k_{z2} (x, y) + f^k_{u_{3z}} (z) u^k_{z3} (x, y) + \ldots
\end{align*}
\]

As an example, a Layer-Wise theory could be created by adopting, for each layer \( k \), a Classical Plate Theory. In other words, the displacement field for each layer could be prescribed as if the layer was a single plate analyzed with the Classical Plate Theory. A theory built by using this technique would be the following:

\[
\begin{align*}
  u^k_x (x, y, z) &= u^k_{x0} (x, y) - \left( z - \frac{z_{\text{top}_k} + z_{\text{bot}_k}}{2} \right) \frac{\partial u^k_{x0} (x, y)}{\partial x} \\
  u^k_y (x, y, z) &= u^k_{y0} (x, y) - \left( z - \frac{z_{\text{top}_k} + z_{\text{bot}_k}}{2} \right) \frac{\partial u^k_{y0} (x, y)}{\partial y} \\
  u^k_z (x, y, z) &= u^k_{z0} (x, y)
\end{align*}
\]

It should be observed that the term \( z - \frac{z_{\text{top}_k} + z_{\text{bot}_k}}{2} \) is a consequence of the fact that if the Classical Plate Theory is used for each layer then the thickness coordinate must be referred to the mid-plane of that layer. Note that equation 20 could be obtained as a particular case of the generic writing reported in equation 19. A different type of LW theory could be obtained for example by selecting a First Order Shear Deformation theory reported in equation 17. Equation 21 is an example of Partially Zig-Zag Higher Order Shear Deformation Theories (PZZHSDTs)

A typical Zig-Zag displacement field for a three-layered structure is reported in Figure 3. It is clear that the discontinuity of the first derivative is more pronounced for the in-plane displacements \( u_x \) and \( u_y \). Thus, the best compromise between the computational cost (more unknowns increase the computational cost) and the accuracy would be to add the MZZFs only for the in-plane displacements. For the baseline theory reported in equation 5, the best compromise would be

\[
\begin{align*}
  u_x (x, y, z) &= u_{x0} (x, y) + z \phi_{1u_x} (x, y) + z^2 \phi_{2u_x} (x, y) + (-1)^k \zeta_k u_{xz} (x, y) \\
  u_y (x, y, z) &= u_{y0} (x, y) + z \phi_{1u_y} (x, y) + z^2 \phi_{2u_y} (x, y) + (-1)^k \zeta_k u_{yz} (x, y) \\
  u_z (x, y, z) &= u_{z0} (x, y)
\end{align*}
\]

instead of equation 17. Equation 21 is an example of Partially Zig-Zag Higher Order Shear Deformation Theory (PZZHSDT). In equation 21 the order of expansion for the in-plane displacements is equal to two plus MZZFs. For more complex cases the order used for the expansion can be increased. Another example of PZZHSDT is given by the following displacement field:

\[
\begin{align*}
  u_x (x, y, z) &= u_{x0} (x, y) + z \phi_{1u_x} (x, y) + z^2 \phi_{2u_x} (x, y) + (-1)^k \zeta_k u_{xz} (x, y) \\
  u_y (x, y, z) &= u_{y0} (x, y) + z \phi_{1u_y} (x, y) + z^2 \phi_{2u_y} (x, y) \\
  u_z (x, y, z) &= u_{z0} (x, y) + (-1)^k \zeta_k u_{zz} (x, y)
\end{align*}
\]
The following definition is used:

A partially Zig-Zag Higher Order Shear Deformation Theory (PZZHSDT) is obtained from an Equivalent Single Layer Theory (called baseline theory) by enhancing some (but not all) displacements with Murakami’s Zig-Zag Function. The baseline theory presents a zero transverse normal strain $\varepsilon_{zz}$. PZZHSDT is also an Equivalent Single Layer Theory.

L. Partially Zig-Zag Advanced Higher Order Shear Deformation Theories (PZZAHSDTs)

In the case of Partially Zig-Zag Advanced Higher Order Shear Deformation Theories (PZZAHSDTs) the baseline theory presents a non-zero transverse normal strain. In particular, the following definition is adopted:

A partially Zig-Zag Advanced Higher Order Shear Deformation Theory (PZZAHSDT) is obtained from an Equivalent Single Layer Theory (called baseline theory) by enhancing some (but not all) displacements with Murakami’s Zig-Zag Function. The baseline theory presents a non-zero transverse normal strain $\varepsilon_{zz}$. PZZAHSDT is also an Equivalent Single Layer Theory.

An example of PZZAHSDT is the following:

$$
\begin{align*}
  u_x(x, y, z) &= u_{x0}(x, y) + z\phi_{1u_x}(x, y) + z^2\phi_{2u_x}(x, y) + (-1)^k \zeta_k u_{xZ}(x, y) \\
  u_y(x, y, z) &= u_{y0}(x, y) + z\phi_{1u_y}(x, y) + z^2\phi_{2u_y}(x, y) \\
  u_z(x, y, z) &= u_{z0}(x, y) + z\phi_{1u_z}(x, y) + (-1)^k \zeta_k u_{zZ}(x, y)
\end{align*}
$$

M. Partially Layer-Wise Higher Order Shear Deformation Theories (PLHSDTs)

If the three-dimensional effects are considered important, some variables may be described in a Layer-Wise sense (i.e., the unknown functions used in the expansion of the displacement variable depend on the layer $k$). An example of PLHSDT is the following:

$$
\begin{align*}
  u^k_x &= u^k_{x0} + \left( z - \frac{z_{topk} + z_{botk}}{2} \right) \phi^k_{1u_x} + \left( z - \frac{z_{topk} + z_{botk}}{2} \right)^2 \phi^k_{2u_x} \\
  u^k_y &= u^k_{y0} + \left( z - \frac{z_{topk} + z_{botk}}{2} \right) \phi^k_{1u_y} + \left( z - \frac{z_{topk} + z_{botk}}{2} \right)^2 \phi^k_{2u_y} \\
  u^k_z &= u^k_{z0}
\end{align*}
$$

In equation 24, the in-plane displacements $u_x$ and $u_y$ are expanded by using a Layer-Wise formulation. In particular, a second order shear deformation theory is used for each layer $k$. The out-of-plane displacement $u_z$ presents an Equivalent Single Layer description. In the case of equation 24 the displacement in the thickness direction is constant.

The following definition is adopted:

Partially Layer-Wise Higher Order Shear Deformation Theory (PLHSDT) presents some quantities described in a Layer-Wise sense and other quantities described in an ESL sense. PLHSDT presents a zero transverse normal strain $\varepsilon_{zz}$ and non-zero transverse shear strains. PLHSDT is neither an ESL theory nor a LW theory since both types of representations are present.

N. Partially Layer-Wise Advanced Higher Order Shear Deformation Theories (PLAHSDTs)

The Partially Layer-Wise Advanced Higher Order Shear Deformation Theories (PLAHSDTs) are very similar to the PLHSDTs. However, the transverse shear strain effects are now retained. A possible example of
PLAHSDT is the following:

\[
\begin{align*}
    u^k_x &= u^k_{x0} + \left( z - \frac{z_{\text{top}_k} + z_{\text{bot}_k}}{2} \right) \phi^k_{1u_x} + \left( z - \frac{z_{\text{top}_k} + z_{\text{bot}_k}}{2} \right)^2 \phi^k_{2u_x} \\
    u^k_y &= u^k_{y0} + \left( z - \frac{z_{\text{top}_k} + z_{\text{bot}_k}}{2} \right) \phi^k_{1u_y} + \left( z - \frac{z_{\text{top}_k} + z_{\text{bot}_k}}{2} \right)^2 \phi^k_{2u_y} \\
    u^k_z &= u^k_{z0} + z\phi^k_{1u_z} + z^2\phi^k_{2u_z} \\
\end{align*}
\]

where the in-plane displacement variables are expanded with a parabolic Layer-Wise Taylor polynomial, whereas the transverse displacement presents a parabolic Equivalent Single Layer description. Another example of PLAHSDT would be the following:

\[
\begin{align*}
    u^k_x &= u^k_{x0} + z\phi^k_{1u_x} \\
    u^k_y &= u^k_{y0} + \left( z - \frac{z_{\text{top}_k} + z_{\text{bot}_k}}{2} \right) \phi^k_{1u_y} + \left( z - \frac{z_{\text{top}_k} + z_{\text{bot}_k}}{2} \right)^2 \phi^k_{2u_y} \\
    u^k_z &= u^k_{z0} + z\phi^k_{1u_z} + z^2\phi^k_{2u_z} \\
\end{align*}
\]

where \( z_{\text{bot}_k} \leq z \leq z_{\text{top}_k} \)

In the theory presented in equation 26, the in-plane displacement \( u_x \) is axiomatically expanded with a linear Equivalent Single Layer description; the in-plane displacement \( u_y \) presents a linear Layer-Wise description and the transverse displacement \( u_z \) is described with a parabolic Layer-Wise expansion. The following definition is adopted:

*Partially Layer-Wise Advanced Higher Order Shear Deformation Theory (PLAHSDT)* presents some quantities described in a Layer-Wise sense and other quantities described in an ESL sense. PLAHSDT presents a non-zero transverse normal strain \( \epsilon_{zz} \) and non-zero transverse shear strains. **PLAHSDT is neither an ESL theory nor a LW theory since both types of representations are present.**

### O. Partially Layer-Wise Zig-Zag and Higher Order Shear Deformation Theories (PLZZHSDTs)

In a displacement-based theory the displacements \( u_x \), \( u_y \), and \( u_z \) are the variables which need to be axiomatically described. It has been shown that the accuracy is higher for Layer-wise theories followed by the Zig-Zag theories and Higher Order Shear Deformation Theories. The Partially Layer-Wise Zig-Zag and Higher Order Shear Deformation Theories (PLZZHSDTs) have the following properties:

- One displacement variable presents a LW description.
- One displacement variable presents an ESL description and is enhanced with Murakami’s Zig-Zag Function.
- One displacement variable presents an ESL description but is not enhanced with Murakami’s Zig-Zag Function.
- The transverse shear strains are not zero
- The transverse normal strain \( \epsilon_{zz} \) is equal to zero

A possible PLZZHSDT is the following:

\[
\begin{align*}
    u^k_x &= u^k_{x0} + \left( z - \frac{z_{\text{top}_k} + z_{\text{bot}_k}}{2} \right) \phi^k_{1u_x} + \left( z - \frac{z_{\text{top}_k} + z_{\text{bot}_k}}{2} \right)^2 \phi^k_{2u_x} \\
    u^k_y &= u^k_{y0} + z\phi^k_{1u_y} + z^2\phi^k_{2u_y} + (-1)^k \zeta_k u^k_{yz} \\
    u^k_z &= u^k_{z0} \\
\end{align*}
\]
In the example reported in equation 27 the displacement $u_x$ presents a parabolic LW description, the displacement $u_y$ presents a parabolic ESL description with Zig-Zag enhancement by the means of Murakami’s Zig-Zag Function and the displacement $u_z$ presents an ESL expansion and is constant.

P. Partially Layer-Wise Advanced Zig-Zag and Higher Order Shear Deformation Theories (PLAZZHSDTs)

Partially Layer-Wise Advanced Zig-Zag and Higher Order Shear Deformation Theories (PLAZZHSDTs) are very similar to the (PLZZHSDTs) previously discussed. The difference is in the transverse normal strain $\epsilon_{zz}$ which is not zero. An example of PLAZZHSDT is the following:

$$
\begin{align*}
\begin{cases}
  u_x^k &= u_{x0}^k + \left( z - \frac{z_{\text{top}_k} + z_{\text{bot}_k}}{2} \right) \phi_{1u_x}^k + \left( z - \frac{z_{\text{top}_k} + z_{\text{bot}_k}}{2} \right)^2 \phi_{2u_x}^k \\
  u_y &= u_{y0} + z\phi_{1u_y} + z^2\phi_{2u_y} \\
  u_z &= u_{z0} + z\phi_{1u_z} + z^2\phi_{2u_z} + (-1)^k \zeta_k u_{zz}
\end{cases}
\end{align*}
$$

In the example reported in equation 28 the displacement $u_x$ presents a parabolic LW description, the displacement $u_y$ presents a parabolic ESL description and the displacement $u_z$ presents a parabolic ESL expansion with Zig-Zag enhancement by the means of Murakami’s Zig-Zag Function. In general, a PLAZZHSDT has the following properties:

- One displacement variable presents a LW description.
- One displacement variable presents an ESL description and is enhanced with Murakami’s Zig-Zag Function.
- One displacement variable presents an ESL description and is not enhanced with Murakami’s Zig-Zag Function.
- The transverse shear strains are not zero.
- The transverse normal strain $\epsilon_{zz}$ is a non-zero quantity.

Q. An Improved Layer-Wise Axiomatic Expansion Based on Combination of Legendre’s Polynomials

The axiomatic expansion adopted for a generic displacement variable does not necessarily have to be a Taylor polynomial function of the thickness coordinate $z$. It can be a set of trigonometric (or other types of) functions. Suppose that the variable $u_x$ presents a Layer-Wise description in the thickness direction. Consider for example the parabolic expansion presented in equation 28:

$$
\begin{align*}
  u_x^k &= u_{x0}^k + \left( z - \frac{z_{\text{top}_k} + z_{\text{bot}_k}}{2} \right) \phi_{1u_x}^k + \left( z - \frac{z_{\text{top}_k} + z_{\text{bot}_k}}{2} \right)^2 \phi_{2u_x}^k
\end{align*}
$$

It is relevant to note that the quantity $u_{x0}^k$ represents the actual displacement on the mid-surface of layer $k$. From equation 29 the displacement at the top and bottom layer’s surfaces (they are indicated as $u_x^{k_t}$ and $u_x^{k_b}$ respectively) can be calculated:

$$
\begin{align*}
  u_x^k \left( z = z_{\text{top}_k} \right) &= u_x^{k_t} = u_{x0}^k + \left( \frac{z_{\text{top}_k} - z_{\text{bot}_k}}{2} \right) \phi_{1u_x}^k + \left( \frac{z_{\text{top}_k} - z_{\text{bot}_k}}{2} \right)^2 \phi_{2u_x}^k \\
  u_x^k \left( z = z_{\text{bot}_k} \right) &= u_x^{k_b} = u_{x0}^k + \left( \frac{z_{\text{bot}_k} - z_{\text{top}_k}}{2} \right) \phi_{1u_x}^k + \left( \frac{z_{\text{bot}_k} - z_{\text{top}_k}}{2} \right)^2 \phi_{2u_x}^k
\end{align*}
$$

From equation 30 it can be observed that:

- The layer’s top surface displacement depends on the combination of more than one degrees of freedom. In the particular example of equation 30 $u_x^{k_t}$ (which is the top surface displacement in the $x$ direction) depends on the following degrees of freedom: $u_{x0}^k$, $\phi_{1u_x}^k$, and $\phi_{2u_x}^k$. 

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Similarly, the layer’s bottom surface displacement depends on the combination of more than one degrees of freedom.

This is consistent with the displacement field reported in equation 29. However, the axiomatic parabolic expansion of equation 29 is not a convenient representation. This can be understood by considering that the compatibility of the displacement \( u_x \) needs to be imposed between two adjacent layers (identified by the identities \( k \) and \( k + 1 \) respectively). This implies \( u_x^{(k)} = u_x^{(k+1)} \) or, using the displacement field reported in equation 29:

\[
\begin{align*}
    u_x^k + \left( \frac{z_{\text{top}_k} - z_{\text{bot}_k}}{2} \right) \phi_{1u_x}^k + \left( \frac{z_{\text{top}_k} - z_{\text{bot}_k}}{2} \right)^2 \phi_{2u_x}^k &= u_x^{(k+1)} + \left( \frac{z_{\text{bot}_{(k+1)}} - z_{\text{top}_k}}{2} \right) \phi_{1u_x}^{(k+1)} + \left( \frac{z_{\text{bot}_k} - z_{\text{top}_{(k+1)}}}{2} \right)^2 \phi_{2u_x}^{(k+1)} \\
    u_x^{(k+1)} + \left( \frac{z_{\text{bot}_{(k+1)}} - z_{\text{top}_{(k+1)}}}{2} \right) \phi_{1u_x}^{(k+1)} + \left( \frac{z_{\text{bot}_k} - z_{\text{top}_{(k+1)}}}{2} \right)^2 \phi_{2u_x}^{(k+1)} &= \phi_{1u_x}^k + \left( \frac{z_{\text{top}_k} - z_{\text{bot}_k}}{2} \right) \phi_{2u_x}^k
\end{align*}
\]

Which is quite complex. In particular, it is deduced the following general statement (valid not only for the parabolic expansion reported in equation 29):

A Taylor polynomial thickness expansion for a displacement variable described in a Layer-Wise sense is not convenient since many degrees of freedom are involved in the writing of the interlaminar compatibility condition.

Ideally the interlaminar compatibility should involve only two degrees of freedom which should take the meaning of physical displacements evaluated exactly at the interface between the two layers \( k \) and \( k + 1 \). This can be achieved if a different axiomatic (but still parabolic) model is adopted (compare equations 32 and 29):

\[
u_x^k = \frac{P_0^k + P_1^k}{2} u_{xz}^k + (P_2^k - P_0^k) u_{xz_b}^k + \frac{P_0^k - P_1^k}{2} u_{xb}^k
\]  

(32)

where the quantities \( P_0^k, P_1^k, \) and \( P_2^k \) are defined as follows:

\[
\begin{align*}
P_0^k &= 1 \\
P_1^k &= \zeta_k \\
P_2^k &= \frac{3 (\zeta_k)^2 - 1}{2}
\end{align*}
\]

(33)

The degrees of freedom which appear in equation 32 are \( u_{xz}^k, u_{xz_b}^k, \) and \( u_{xb}^k \). Please note that they are not coincident to the ones reported in the original parabolic representation based on Taylor polynomials (see equation 29). None of them represents the displacement in the mid-surface of layer \( k \). To see what the degrees of freedom represent, let one evaluate the terms of equation 33 at the thickness coordinate \( z \) corresponding to the top surface of layer \( k \): \( z = z_{\text{top}_k} \). It is possible to write:

\[
\begin{align*}
    \zeta_k (z = z_{\text{top}_k}) &= \frac{2}{z_{\text{top}_k} - z_{\text{bot}_k}} z_{\text{top}_k} - z_{\text{bot}_k} = 1 \\
P_0^k (z = z_{\text{top}_k}) &= 1 \\
P_1^k (z = z_{\text{top}_k}) &= \zeta_k (z = z_{\text{top}_k}) = 1 \\
P_2^k (z = z_{\text{top}_k}) &= \frac{3}{2} (\zeta_k (z = z_{\text{top}_k})^2 - 1) = 1
\end{align*}
\]

(34)

The displacement evaluated at the top surface of layer \( k \) (indicated as \( u_{x}^{k,t} \)) is obtained by combining equations 34 and 32:

\[
u_x^k (z = z_{\text{top}_k}) = u_{x}^{k,t} = \frac{1}{2} u_{x}^k + (1 - 1) u_{xz}^k + \frac{1}{2} u_{xb}^k = u_{x}^k
\]

(35)

Thus, it is deduced that

The degree of freedom \( u_{x}^k \) takes the meaning of displacement in the \( x \) direction calculated at the top surface
of layer \( k \).

Using a similar procedure, it can be shown that

The degree of freedom \( u^k_{x_b} \) takes the meaning of displacement in the \( x \) direction calculated at the bottom surface of layer \( k \).

It is then clear that the axiomatic expansion used in equation 32 should be preferred to the alternative representation reported in equation 29. In fact, instead of the complicate relation equation 31, which is valid for the Taylor expansion reported in equation 29, now the interlaminar compatibility would just be written as

\[
\begin{align*}
\frac{u^k}{x} \left( \zeta = z_{\text{top}_k} \right) &= \frac{u^{k+1}}{x} \left( \zeta = z_{\text{bot}_{(k+1)} } \right) \Rightarrow u^k_{x_i} = u^{k+1}_{x_i}
\end{align*}
\]

which is extremely simple. This is particularly advantageous in the assembling of the matrices in the thickness direction, as will be discussed when the Generalized Unified Formulation is introduced. Is it possible to generalize the strategy developed for the parabolic case (see equation 29 for the Taylor representation and its improvement as seen in equation 32 which led to particularly simple expression for the interlaminar compatibility condition) and analyze situations in which the orders of expansions are different than the parabolic? The answer is yes. Suppose for example, that the desired order for the expansion of the displacement \( u \) is \( N_{u_x} = N_{u_z} \). The superscript in \( N_{u_x} \) is removed because it is assumed that the order of expansion is the same in each layer. In the case of equation 32 it was \( N_{u_z} = 2 \). The counterpart of equation 32 is now the following:

\[
\begin{align*}
\frac{u^k}{x} & = \frac{P^k_0 + P^k_1}{2} u^k_{x_i} + (P^k_2 - P^k_0) u^k_{x_2} + (P^k_3 - P^k_1) u^k_{x_3} + ... \\
& + \left( P^k_{N_{u_x}} - P^k_{N_{u_x} - 2} \right) u^k_{x_N_{u_x}} + \frac{P^k_0 - P^k_1}{2} u^k_{x_b}
\end{align*}
\]

where the terms \( P^k_i \) are Legendre’s polynomials of order \( i \) evaluated in \( \zeta_k \). For example, \( P^k_3 \) is the cubic Legendre’s polynomial. As shown for the parabolic case, in equation 37 \( u^k_{x} \) represents the layer’s top surface displacement and \( u^k_{x_b} \) represents the layer’s bottom surface displacement. This is the case because the following statements are always true:

\[
\zeta_k = \begin{cases} 
+1, & \frac{P^k_0 + P^k_1}{2} = 1, \\
-1, & \frac{P^k_0 + P^k_1}{2} = 0
\end{cases} \quad \begin{cases} 
(P^k_2 - P^k_1 - 2) = 0, \\
(P^k_1 - P^k_1 - 2) = 0
\end{cases}
\]

Equation 37 is conveniently written in a more compact form as follows:

\[
\frac{u^k}{x} = \sum_{t} x^k_{F_t} u^k_{x_t} + \sum_{t=2} x^k_{F_2} u^k_{x_2} + \sum_{t=3} x^k_{F_3} u^k_{x_3} + ... + \sum_{t=N_{u_x}} x^k_{F_{N_{u_x}}} u^k_{x_{N_{u_x}}} + \sum_{t=N_{u_x}} x^k_{F_{N_{u_x}}} u^k_{x_{N_{u_x}}}
\]

where (see equation 37)

\[
\begin{align*}
\sum_{t} x^k_{F_t} &= \frac{P^k_0 + P^k_1}{2} \\
\sum_{t=2} x^k_{F_2} &= P^k_2 - P^k_0 \\
\sum_{t=3} x^k_{F_3} &= P^k_3 - P^k_1 \\
&\vdots \\
\sum_{t=N_{u_x}} x^k_{F_{N_{u_x}}} &= P^k_{N_{u_x}} - P^k_{N_{u_x} - 2} \\
\sum_{t=N_{u_x}} x^k_{F_{N_{u_x}}} &= P^k_{N_{u_x}} - P^k_{N_{u_x} - 2} \frac{P^k_0 - P^k_1}{2}
\end{align*}
\]

Legendre’s polynomials could also be used for the Equivalent Single Layer case with no significant complication of the formulation. However, to keep the consistency with a large amount of literature devoted to HSDT’s, Taylor polynomials will be adopted for the ESL case and Legendre’s polynomials will be employed for the Layer-Wise case. Even if Legendre’s polynomials are used instead of Taylor’s ones, the terminology
previously introduced (e.g., PLAHDSTs etc.) is maintained. It should be noted that the Layer-Wise representation based on Legendre’s polynomials assumes that at least the terms \( u_x^k \) and \( u_z^k \) are present. This corresponds to the linear case. In other words, this very effective axiomatic representation is valid if a linear or higher order expansion is adopted. Another important fact is that with this representation based on Legendre’s polynomials the number of degrees of freedom per layer is equal to order of polynomial incremented or higher order expansion is adopted. Another important fact is that with this representation based on Legendre’s polynomials assumes that at least the terms \( u_x^k \) and \( u_z^k \) are present. In the parabolic case \( (N_{u_x} = 2) \) the number of degrees of freedom and the generalized coordinates are represented by \( u_x^k, u_z^k, \) and \( u_z^k \). In general, for a Legendre's polynomial type of Layer-Wise expansion it is \( N_{OF} u_x^k = N_{u_x} + 1 \). It is obvious that the same arguments could be easily adopted if other displacements (e.g., \( u_y \)) are described in a Layer-Wise sense.

### III. Designation of the Axiomatic Models

The Generalized Unified Formulation is extended for the first time in this work to cover the types of theories previously discussed. The theories are differentiated from each other by the following descriptors:

- **Variational statement**: the governing equations could be variationally derived from a displacement-based statement such as the Principle of Virtual Displacements (PVD) or from a mixed statement such as Reissner Mixed Variational Theorem (RMVT). In this work the focus is only on PVD-based theories: only the displacements are axiomatically described.

- **Order of expansion**: the different displacement variables can have a freely and independent order of expansion in the thickness direction.

- **Type of axiomatic model**: the different displacement variables can have an ESL description (enhanced or not with the Zig-Zag function) or a Layer Wise representation.

These descriptors are shown in Figure 4. It should be noted that the class of acronyms defined in Figure 4 is more general than previous designations adopted in past work. For example it is \( ZZZPVD_{213} \equiv EZZ_{213} \), \( ZZEPVD_{213} \not= EDZ_{213} \), and \( EEEPVD_{213} \equiv ED_{213} \). Figure 5 shows some theories and relative notation in the GUF framework.

### IV. Finite Element Discretization in the Framework of the Generalized Unified Formulation

#### A. Axiomatic Expressions within GUF Formalism

To explain how the Finite Element implementation should be carried out in the Generalized Unified Formulation framework, consider for example the displacement variable \( u_x \). Assume that it is represented with a cubic Equivalent Single Layer discretization. It should be noted that for example the theories \( EEEPVD_{333} \), \( ELZPVD_{324} \), \( EZZPVD_{332} \), and \( ELLPVD_{312} \) have this property and they would have identical treatment as far as the displacement \( u_x \) is concerned. The differences among the above mentioned theories would be only in the assembling of the matrices in the thickness direction. Within GUF formalism the displacement \( u_x \) (which is a function of the thickness coordinate \( z \) and the in-plane coordinates \( x, y \)) is written as follows:

\[
\begin{align*}
\quad u_x &= \sum_{\alpha_x - t,b}^{x} F_{\alpha_x u_x z_{\alpha_x}} = \sum_{\alpha_x - t,b}^{x} F_{\alpha_x} u_{z_{\alpha_x}} = \sum_{\alpha_x - t,b}^{x} F_{\alpha_x} u_{z_{\alpha_x}} \\
 &= \alpha_x (x, y) + z\phi_{1_u_x} (x, y) + z^2\phi_{2_u_x} (x, y) + z^3\phi_{3_u_x} (x, y)
\end{align*}
\]  

(41)

The unknown functions \( u_{x0} (x, y) \), \( \phi_{1_u_x} (x, y) \), \( \phi_{2_u_x} (x, y) \), and \( \phi_{3_u_x} (x, y) \) in equation 41 need to be discretized according to the Finite Element Method. Consider, for example, the function \( \phi_{2_u_x} (x, y) \) and its FE representation. Suppose that a 4-node quadrilateral element \((Q4)\) is considered. In that case we have:

\[
\begin{align*}
\phi_{2_u_x} (x, y) &= \sum_{\alpha_x - t,b}^{x} N_1 (x, y) \Phi_{2_u_x,1} + \sum_{\alpha_x - t,b}^{x} N_2 (x, y) \Phi_{2_u_x,2} \\
&+ \sum_{\alpha_x - t,b}^{x} N_3 (x, y) \Phi_{2_u_x,3} + \sum_{\alpha_x - t,b}^{x} N_4 (x, y) \Phi_{2_u_x,4}
\end{align*}
\]  

(42)
Where \( z^x N_1 (x, y) \), \( z^x N_2 (x, y) \), \( z^x N_3 (x, y) \), and \( z^x N_4 (x, y) \) represent the known shape functions used to reconstruct the function \( \phi_{2ux} (x, y) \) at element level. Note that the superscript \( x \) is used in the symbolic representation of the shape functions to emphasize that one term, \( \phi_{2ux} (x, y) \), of the expansion of the displacement in the \( x \) direction, \( u_x \), is considered. The terms \( \Phi_{2ux,1} \), \( \Phi_{2ux,2} \), \( \Phi_{2ux,3} \), and \( \Phi_{2ux,4} \) in equation 42 represent the nodal values of the unknown function \( \phi_{2ux} (x, y) \). Equation 42 can be written in a compact form using the indicial notation:

\[
\phi_{2ux} (x, y) = \sum_{i=1,2,3,4} {z^x} N_i (x, y) \Phi_{2ux,i} \tag{43}
\]

Equation 43 can be put in a form independent of the type of finite element and number of nodes \( N_n \) (which can be a general number) as follows:

\[
\phi_{2ux} (x, y) = \sum_{i=1,\ldots,N_n} z^x N_i (x, y) \Phi_{2ux,i} \tag{44}
\]

Using the summation convention (Einstein’s notation) equation 44 can be written as

\[
\phi_{2ux} (x, y) = z^x N_i (x, y) \Phi_{2ux,i} \quad i = 1, 2, \ldots, N_n \tag{45}
\]

The methodology that led to the writing of equation 45 for the Finite Element representation of \( \phi_{2ux} (x, y) \) can be used for the remaining variables \( u_{x0} (x, y) \), \( \phi_{1ux} (x, y) \), and \( \phi_{3ux} (x, y) \) used in the expansion of the displacement \( u_x \) (see equation 41) and expressions similar to equation 45 are obtained:

\[
\begin{align*}
\phi_{1ux} (x, y) &= z^x N_i (x, y) \Phi_{1ux,i} \\
\phi_{3ux} (x, y) &= z^x N_i (x, y) \Phi_{3ux,i}
\end{align*} \quad i = 1, 2, \ldots, N_n \tag{46}
\]
Example: $ZEE^{PVD}$

<table>
<thead>
<tr>
<th>$u_x$: $N_{ux} = 1$</th>
<th>ESL + Zig-Zag</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_y$: $N_{uy} = 3$</td>
<td>ESL</td>
</tr>
<tr>
<td>$u_z$: $N_{uz} = 2$</td>
<td>ESL</td>
</tr>
</tbody>
</table>

**GUF representation**

- $u_x = f_{a_0} u_{a_0}$
- $u_y = f_{a_0} u_{a_0}$
- $u_z = f_{a_0} u_{a_0}$

**Expansion ($\alpha_{a_0}$, $\alpha_{a_0}$, and $\alpha_{a_0}$)**

- $u_x = F_{a_0} + F_{a_2} + F_{a_3}$
- $u_y = F_{a_0} + F_{a_2} + F_{a_3}$
- $u_z = F_{a_0} + F_{a_2} + F_{a_3}$

**Final Partially Zig-Zag theory**

- $u_x = u_{a_0} + z\Phi_{a_0}$
- $u_y = u_{a_0} + z\Phi_{a_0}$
- $u_z = u_{a_0} + z\Phi_{a_0}$

Figure 5. Examples of possible theories and GUF representation.

where $U_{a_0}$, $\Phi_{a_0}$, and $\Phi_{a_0}$ are the nodal values. Now equations 46 and 45 are written with the GUF notation reported in Equation 41. For that purpose the nodal values need to be renamed as follows:

\[
\begin{align*}
  u_{a_0} &= xN_i U_{a_0} \quad \iff \quad u_{a_i} = xN_i xU_{a_i} \quad i = 1, 2, \ldots, N_n \\
  \phi_{a_0} &= xN_i \Phi_{a_0} \quad \iff \quad \phi_{a_i} = xN_i xU_{a_i} \quad i = 1, 2, \ldots, N_n \\
  \phi_{a_2} &= xN_i \Phi_{a_2} \quad \iff \quad \phi_{a_i} = xN_i xU_{a_i} \quad i = 1, 2, \ldots, N_n \\
  \phi_{a_3} &= xN_i \Phi_{a_3} \quad \iff \quad \phi_{a_i} = xN_i xU_{a_i} \quad i = 1, 2, \ldots, N_n 
\end{align*}
\]  

(47)

where the nodal values have been renamed, according to GUF, as

\[
\begin{align*}
  U_{a_0} &\quad \rightarrow xU_{a_i} \\
  \Phi_{a_0} &\quad \rightarrow xU_{a_i} \\
  \Phi_{a_2} &\quad \rightarrow xU_{a_i} \\
  \Phi_{a_3} &\quad \rightarrow xU_{a_i}
\end{align*}
\]  

(48)

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The displacement \( u_x \) within the generic Finite Element is then written as follows (see equations 41 and 47):

\[
\begin{align*}
    u_x &= u_{x0} + z \phi_{1x} + z^2 \phi_{2x} + z^3 \phi_{3x} \\
    &= \delta F_1 u_{x1} + \delta F_2 u_{x2} + \delta F_3 u_{x3} + \delta F_b u_{x_b} \\
    &= \delta N_1 \delta U_{1x} + \delta F_2 \delta N_1 \delta U_{2x} + \delta F_3 \delta N_1 \delta U_{3x} + \delta F_b \delta N_1 \delta U_{b1} \\
    &= \delta N_1 (\delta F_1 \delta U_{1x} + \delta F_2 \delta U_{2x} + \delta F_3 \delta U_{3x} + \delta F_b \delta U_{b1}) \\
    &= \delta N_1 \delta F_{\alpha_{ux}} \delta U_{\alpha_{ux}i} = \delta F_{\alpha_{ux}} \delta N_1 \delta U_{\alpha_{ux}i}
\end{align*}
\]

where the indices of the summations of equation 49 vary according to the formula

\[
\alpha_{ux} = t, l, b, \quad l = 2, \ldots, N_{ux} \quad i = 1, 2, \ldots, N_n
\]  

(50)

In the case of cubic expansion for the displacement \( u_x \) and Q4 FE, it is \( N_{ux} = 3 \) and \( N_n = 4 \). However, it is important to notice that equation 49 is formally invariant with respect to the order of expansion (\( N_{ux} \) can assume any integer value) and the type of element (\( N_n \) is another free parameter which changes if another type of finite element is used). The logic used in the derivation of 49 can be applied for all the displacements. Therefore, the Finite Element representation of the displacements for a generic RMVT-based Higher Order Shear Deformation Theory in the framework of the Generalized Unified Formulation is

\[
\begin{align*}
    u_x &= \delta F_{\alpha_{ux}} \delta N_1 \delta U_{\alpha_{ux}i} \quad \alpha_{ux} = t, l, b, \quad l = 2, \ldots, N_{ux} \quad i = 1, 2, \ldots, N_n \\
    u_y &= \delta F_{\alpha_{uy}} \delta N_1 \delta U_{\alpha_{uy}i} \quad \alpha_{uy} = t, m, b, \quad m = 2, \ldots, N_{uy} \quad i = 1, 2, \ldots, N_n \\
    u_z &= \delta F_{\alpha_{uz}} \delta N_1 \delta U_{\alpha_{uz}i} \quad \alpha_{uz} = t, n, b, \quad n = 2, \ldots, N_{uz} \quad i = 1, 2, \ldots, N_n
\end{align*}
\]

(51)

If a variable has a layerwise description it is sufficient to add the superscript \( k \) in the definitions reported in equation 51. Actually, in the GUF formalism the superscript \( k \) is always added so that the layer stiffness matrix generation is formally invariant with respect to the type of representation (i.e., ESL with or without Zig-Zag term or Layerwise) so that the final form which can include all types of axiomatic theories previously listed is the following:

\[
\begin{align*}
    u_x^k &= \delta F_{\alpha_{ux}} \delta N_1 \delta U_{\alpha_{ux}i} \quad \alpha_{ux} = t, l, b, \quad l = 2, \ldots, N_{ux} \quad i = 1, 2, \ldots, N_n \\
    u_y^k &= \delta F_{\alpha_{uy}} \delta N_1 \delta U_{\alpha_{uy}i} \quad \alpha_{uy} = t, m, b, \quad m = 2, \ldots, N_{uy} \quad i = 1, 2, \ldots, N_n \\
    u_z^k &= \delta F_{\alpha_{uz}} \delta N_1 \delta U_{\alpha_{uz}i} \quad \alpha_{uz} = t, n, b, \quad n = 2, \ldots, N_{uz} \quad i = 1, 2, \ldots, N_n
\end{align*}
\]

(52)

The orders of expansions for the a generic variable is assumed to be the same for all layers when a Layer Wise approach is adopted. Moreover, the same type of functions are used in the thickness expansions and this means that no superscript \( k \) is necessary for example for the term \( \delta F_{\alpha_{ux}} \).

Equation 52 is then used to express the strains and their virtual variations in order to apply the Principle of Virtual Displacements. For example, the strain \( \varepsilon_{yy}^k \) can be written, using the invariant property of GUF with respect to the order of expansion:

\[
\varepsilon_{yy}^k = \frac{\partial u_y^k}{\partial y} = \frac{\partial}{\partial y} \left( \delta F_{\alpha_{uy}} \delta N_1 \delta U_{\alpha_{uy}i} \right) = \delta F_{\alpha_{uy}} \frac{\partial \delta N_1}{\partial y} \delta U_{\alpha_{uy}i} = \delta F_{\alpha_{uy}} \delta N_1 \delta U_{\alpha_{uy}i}
\]

(53)

where it has been used the fact that the functions \( \delta F_{\alpha_{uy}} \) depend only on the thickness coordinate \( z \) and the only in-plane variability is due to the shape functions \( \delta N_1 \). \( \delta N_1 \) indicates the derivative with respect to \( y \) of the generic shape function \( \delta N_i \).

The virtual strains (used later in the writing of the Principle of Virtual Work) can be immediately obtained from equation 53 by direct application of the variational operator \( \delta \):

\[
\delta \varepsilon_{yy}^k = \delta F_{\alpha_{uy}} \delta N_1 \delta U_{\alpha_{uy}i}
\]

(54)
For completeness the transverse shear strain $\gamma_{zx}^k$ is also presented:

$$
\gamma_{zx}^k = \frac{\partial u_z^k}{\partial x} + \frac{\partial u_x^k}{\partial z} = \frac{\partial}{\partial x} \left( z F_{\alpha u_z}^N_{i, x} U_{\alpha u_z}^k \right) + \frac{\partial}{\partial z} \left( z F_{\alpha u_x}^N_{i, z} U_{\alpha u_x}^k \right)
$$

(B. Governing Equations within GUF Formalism)

The internal virtual work at layer level is indicated with $\delta W^k_l$. Its expression involves the virtual variations of the strains and the stresses.

$$
\delta W^k_l = \int_{V^k} \left( \delta \epsilon^{k}_{xx} \sigma^{k}_{xx} + \delta \epsilon^{k}_{yy} \sigma^{k}_{yy} + \delta \gamma^{k}_{zy} \sigma^{k}_{zy} + \delta \gamma^{k}_{zx} \sigma^{k}_{zx} + \delta \gamma^{k}_{xz} \sigma^{k}_{xz} + \delta \epsilon^{k}_{zz} \sigma^{k}_{zz} \right) dV^k
$$

where $V^k$ indicates the volume of layer $k$.

One of the key features of the Generalized Unified Formulation is the writing of all the equations in a compact form. Using the geometric relations to express the strains as function of the displacements' derivatives (see for example the derivation presented in equation 53) and adopting Hooke’s law to express the stresses as a function of the strains (so that the remaining unknowns are just the displacements) the internal virtual work at layer level can be demonstrated to be

$$
\delta W^k_l = \delta U^{k}_{\alpha u_u} i K_{u \alpha u u}^{k \alpha u_u \beta u_u} ij x U^{k}_{\beta u_u j} + \delta U^{k}_{\alpha u_y} i K_{u \alpha u y}^{k \alpha u_y \beta u_y} ij y U^{k}_{\beta u_y j} + \delta U^{k}_{\alpha u_z} i K_{u \alpha u z}^{k \alpha u_z \beta u_z} ij z U^{k}_{\beta u_z j} + \delta U^{k}_{\alpha u_x} i K_{u \alpha u x}^{k \alpha u_x \beta u_x} ij x U^{k}_{\beta u_x j} + \delta U^{k}_{\alpha u_y} i K_{u \alpha u y}^{k \alpha u_y \beta u_y} ij y U^{k}_{\beta u_y j} + \delta U^{k}_{\alpha u_z} i K_{u \alpha u z}^{k \alpha u_z \beta u_z} ij z U^{k}_{\beta u_z j} + \delta U^{k}_{\alpha u_x} i K_{u \alpha u x}^{k \alpha u_x \beta u_x} ij x U^{k}_{\beta u_x j} + \delta U^{k}_{\alpha u_y} i K_{u \alpha u y}^{k \alpha u_y \beta u_y} ij y U^{k}_{\beta u_y j} + \delta U^{k}_{\alpha u_z} i K_{u \alpha u z}^{k \alpha u_z \beta u_z} ij z U^{k}_{\beta u_z j}
$$

(57)

where for example the term $K_{u \alpha u u}^{k \alpha u_u \beta u_u} ij$ is the kernel of the Generalized Unified Formulation and is used to generate a portion of the structural stiffness matrix. This is accomplished by expanding the indices as explained in Figure 6. Equating the internal virtual work and the external virtual work leads to the following

__Figure 6. Definition of one of the invariant kernel: case of $K_{u \alpha u}^{k \alpha u u \beta u_u \beta u_u} ij$__
set of equations:
\[
+ \delta x^{k} U_{\alpha z i}^{k} K_{ux u_x}^{k} \alpha z \beta_{z i j} x U_{\beta u_x}^{k} + \delta y^{k} U_{\alpha z i}^{k} K_{uy u_y}^{k} \alpha z \beta_{u y i j} y U_{\beta u_y}^{k} + \delta z^{k} U_{\alpha z i}^{k} K_{uz u_z}^{k} \alpha z \beta_{u z i j} z U_{\beta u_z}^{k} = \delta x^{k} U_{\alpha z i}^{k} \beta_{z i j} x P_{\alpha z i}^{k} + \delta y^{k} U_{\alpha z i}^{k} \beta_{y i j} y P_{\alpha z i}^{k} + \delta z^{k} U_{\alpha z i}^{k} \beta_{z i j} z P_{\alpha z i}^{k}
\]
(58)
\[
+ \delta y^{k} U_{\alpha y i}^{k} K_{uy u_y}^{k} \alpha y \beta_{y i j} y U_{\beta u_y}^{k} + \delta y^{k} U_{\alpha y i}^{k} K_{uy u_y}^{k} \alpha y \beta_{y i j} y U_{\beta u_y}^{k} + \delta z^{k} U_{\alpha y i}^{k} K_{uy u_y}^{k} \alpha y \beta_{y i j} z U_{\beta u_z}^{k} = \delta y^{k} U_{\alpha y i}^{k} \beta_{y i j} y P_{\alpha y i}^{k} + \delta z^{k} U_{\alpha y i}^{k} \beta_{y i j} z P_{\alpha y i}^{k}
\]
(59)
\[
+ \delta z^{k} U_{\alpha z i}^{k} K_{uz u_z}^{k} \alpha z \beta_{z i j} z U_{\beta u_z}^{k} = \delta z^{k} U_{\alpha z i}^{k} \beta_{z i j} z P_{\alpha z i}^{k}
\]
(60)
The indices are expanded and the arrays are assembled in the thickness direction as explained in the conceptual sketch reported in Figure 6. For example, after the thickness assembling equation 58 reads as follows:
\[
\delta x^{U T} (K_{ux u_x} x U + K_{ux u_y} y U + K_{ux u_z} z U - x P) = 0
\]
(61)
where the vectors \(x U\), \(y U\), and \(z U\) contain the nodal displacements in the \(x\), \(y\), and \(z\) directions at element level. \(x P\) is a vector whose components are the equivalent nodal forces in the \(x\) direction at element level. The components of vector \(\delta x U\) are the virtual variations of the nodal displacements in the \(x\) direction at element level. The superscript \(T\) indicates the transpose operator. From the definition of virtual variations it can be inferred that equation 61 is satisfied if the following relation holds:
\[
K_{ux u_x} x U + K_{ux u_y} y U + K_{ux u_z} z U - x P = 0
\]
(62)
which represents the first set of governing equations. The procedure can be repeated for the other relations (see equations 59 and 60). It is possible to demonstrate that the complete set of governing equations at finite element level is
\[
K_{UU} \cdot U = P
\]
(63)
where \(U\) contains all the vectors of nodal displacements at element level and \(P\) contains all the vectors of nodal forces at element level. The explicit expressions for the vectors of unknowns nodal displacements and known nodal loads (see equation 63) are:
\[
U = \begin{bmatrix} x U \\ y U \\ z U \end{bmatrix}, \quad P = \begin{bmatrix} x P \\ y P \\ z P \end{bmatrix}
\]
(64)
The symmetric finite element stiffness matrix \(K_{UU}\) (see equation 63) is:
\[
K_{UU} = \begin{bmatrix} K_{ux u_x} & K_{ux u_y} & K_{ux u_z} \\ K_{ux u_y}^T & K_{uy u_y} & K_{uy u_z} \\ K_{ux u_z}^T & K_{uy u_z}^T & K_{uz u_z} \end{bmatrix}
\]
(65)

C. Theory-Invariant Arrays: the Kernels of the Generalized Unified Formulation

To write the stiffness matrix \(K_{UU}\) it is required to obtain the size finite element matrices \(K_{ux u_x}, K_{ux u_y}, K_{ux u_z}, K_{uy u_y}, K_{uy u_z},\) and \(K_{uz u_z}\). This means that the actual GUF’s kernels required to generate the stiffness matrix are the following: \(K_{ux u_x}^{k \alpha u_z \beta_{z i j}}\), \(K_{ux u_y}^{k \alpha u_y \beta_{y i j}}\), \(K_{ux u_z}^{k \alpha u_z \beta_{z i j}}\), \(K_{uy u_y}^{k \alpha u_y \beta_{y i j}}\), \(K_{uy u_z}^{k \alpha u_y \beta_{y i j}}\), \(K_{uy u_z}^{k \alpha u_z \beta_{z i j}}\), \(K_{uz u_z}^{k \alpha u_z \beta_{z i j}}\). The explicit expressions involve the thickness integrals of the functions used in the axiomatic expansions. Figure 7 shows two examples of thickness integrals generated for different values of \(\alpha_{u_z}\) and \(\beta_{u_z}\). The explicit expressions for the six kernels are reported in Equations 66 and 67:
Figure 7. Integrals along the thickness: definitions.

\[ K_{u_x u_z}^{k \alpha_{u_x} \beta_{u_z} ij} = \int_{\Omega} Z_{13}^{k \alpha_{u_x} \beta_{u_z} x} N_{i,x} \, x \, N_{j,y} \, dxdy + \int_{\Omega} Z_{13}^{k \alpha_{u_x} \beta_{u_z} x} N_{i,x} \, y \, N_{j,y} \, dxdy + \int_{\Omega} Z_{16}^{k \alpha_{u_x} \beta_{u_z} x} N_{i,y} \, x \, N_{j,y} \, dxdy + \int_{\Omega} Z_{26}^{k \alpha_{u_x} \beta_{u_z} y} N_{i,y} \, y \, N_{j,y} \, dxdy + \int_{\Omega} Z_{55}^{k \alpha_{u_x} \beta_{u_z} x} N_{i,y} \, y \, N_{j,y} \, dxdy \]

\[ K_{u_x u_y}^{k \alpha_{u_x} \beta_{u_y} ij} = \int_{\Omega} Z_{12}^{k \alpha_{u_x} \beta_{u_y} x} N_{i,x} \, x \, N_{j,y} \, dxdy + \int_{\Omega} Z_{12}^{k \alpha_{u_x} \beta_{u_y} x} N_{i,x} \, y \, N_{j,y} \, dxdy + \int_{\Omega} Z_{26}^{k \alpha_{u_x} \beta_{u_y} y} N_{i,y} \, x \, N_{j,y} \, dxdy + \int_{\Omega} Z_{45}^{k \alpha_{u_x} \beta_{u_y} y} N_{i,y} \, y \, N_{j,y} \, dxdy \]

\[ K_{u_x u_z}^{k \alpha_{u_x} \beta_{u_z} ij} = \int_{\Omega} Z_{13}^{k \alpha_{u_x} \beta_{u_z} x} N_{i,z} \, x \, N_{j,z} \, dxdy + \int_{\Omega} Z_{13}^{k \alpha_{u_x} \beta_{u_z} x} N_{i,z} \, z \, N_{j,z} \, dxdy + \int_{\Omega} Z_{36}^{k \alpha_{u_x} \beta_{u_z} x} N_{i,z} \, x \, N_{j,y} \, dxdy + \int_{\Omega} Z_{45}^{k \alpha_{u_x} \beta_{u_z} x} N_{i,z} \, z \, N_{j,z} \, dxdy \]
\[ K^{k \alpha u_y \beta u_y} = \int_{\Omega} Z^{k \alpha u_y \beta u_y} yN_{i,y} yN_{j,y} \, dx\,dy + \int_{\Omega} Z^{k \alpha u_y \beta u_y} yN_{i,j} yN_{j,x} \, dx\,dy \]
\[ + \int_{\Omega} Z^{k \alpha u_y \beta u_y} yN_{i,z} yN_{j,x} \, dx\,dy \]
\[ + \int_{\Omega} Z^{k \alpha u_y \beta u_y} yN_{i}{y} yN_{j} \, dx\,dy. \]

\[ K^{k \alpha u_x \beta u_x} = \int_{\Omega} Z^{k \alpha u_x \beta u_x} zN_{i,x} zN_{j,y} \, dx\,dy + \int_{\Omega} Z^{k \alpha u_x \beta u_x} zN_{i,j} zN_{j,x} \, dx\,dy \]
\[ + \int_{\Omega} Z^{k \alpha u_x \beta u_x} zN_{i,x} zN_{j,x} \, dx\,dy \]
\[ + \int_{\Omega} Z^{k \alpha u_x \beta u_x} zN_{i}{x} zN_{j} \, dx\,dy. \]

(67)

where the framed terms are the ones that need to be integrated in a reduced manner to remove the locking\textsuperscript{60, 61} mechanism. Other more efficient numerical techniques could be implemented in commercial codes based on GUF.

**D. Assembling of the Matrices in the Thickness Direction**

The assembling in the thickness direction need to take into account how the single displacement variable is axiomatically postulated. This is particularly crucial for the stiffness terms which couple different displacement variables (e.g., \( K^{k \alpha u_x \beta u_x} \)). Figures 8-13 show some examples involving the assembling in the thickness direction. The assembly in the plate’s plane is performed according to the standard practice typical of FE codes and the details are omitted for brevity. After the assembling is completed the FE equations at structural assembled level are the following:

\[ K^S_{\bar{U} \bar{U}} \cdot \bar{U}^S = \bar{P}^S \]

(68)

where the superscript \( S \) is adopted to indicate that the stiffness matrix, nodal displacements and nodal forces are evaluated at global structural level.
Figure 8. Example of a finite element matrix at layer level. Case of matrix $K^k_{ux, uy}$ and $Q^4$ finite element.

In this example:

$$ELZ_P V D_{232} \Rightarrow N_{a} = 2 \text{ and } N_{b} = 3$$

$$\alpha_{ux} = i, l, b \quad l = 2$$

$$\beta_{uy} = i, m, b \quad m = 2, 3$$

Matrix at layer and nodal levels

Figure 9. Example of a finite element matrix at layer and nodal levels. Case of matrix $K^k_{ux, uy}$, $Q^4$ finite element, and theory $ELZ_P V D_{232}$.

It is assumed $z_{bot} = -h/2$ and $z_{top} = 0$

$$z_{top} = \int_{z_{bot}}^{z_{top}} \int F_1 F_2 \, dz = -h/2 C_{12}$$

$$\bar{Z}_{12}^{k2} = C_{12}^k$$

$$x_{F_1} = 1$$

$$\int_{x_{F_1}}^{x_{F_2}} = P_2 - P_0$$

$$K^k_{ux, uy} = \int_{\Omega} Z_{12}^{k2} N_{1,x} N_{4,y} \, dx dy + \ldots \text{ other terms}$$

Figure 10. Example of calculation of the kernel $K^k_{ux, uy}^{i,j}$ for particular values of the indices ($\alpha_{ux} = i$, $\beta_{uy} = 2$, $i = 1$, and $j = 4$). Case of kernel $K^k_{ux, uy}^{i,j}$. 

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Figure 11. Example of matrix at multilayer and element levels for a two-layered structure. Case of matrix $K_{u_x u_y}$, $Q4$ finite element, and theory $ELZPVD_{232}$.

Figure 12. Example of matrix at multilayer and node levels for a two-layered structure. Case of matrix $K_{u_x u_y}$, $Q4$ finite element, and theory $ELZPVD_{232}$.

Compatibility of the displacements $u_z$ and $u_y$ are enforced with the assembling along the thickness.

Matrix at multilayer and nodal levels
(In this example 2 layers are considered)
Figure 13. Example of matrix at multilayer and node levels for a two-layered structure. Case of matrix $K_{u_y u_y}^{14}$, $Q4$ finite element, and theory $ELZ PVD_{232}$ (note that the displacement $u_y$ presents a cubic Layer Wise description).

In this example:

$$\Rightarrow N_{by} = 3$$

Matrix at **multilayer** and **nodal** levels

(In this example 2 layers are considered)
V. Description of the Analyzed Test Cases

Figures 14, 15, 16 present the analyzed test cases. Test case 1 (see Figure 14) is a square thick simply supported panel. This test case is used to assess the effect of using an ESL or Zig-Zag-enhanced ESL or LW description for one or more displacement variables (see Figures 17-36 for the results). Test case 2 (see Figure 15) is used to assess the accuracy of the present GUF FEM capability for a more general set of boundary conditions (see Table 1). Finally, the test case 3 (see Figure 16 and Reference [59]) represents a known numerically challenging case with a very high Face-to-Core Stiffness Ratio (FCSR). Test case 3 is adopted to assess when the different displacement variables need a Layer Wise description and the ESL approach (even with the Zig-Zag enhancement) is no longer sufficient for an accurate analysis (see Figures 37 and 38).

**Figure 14. Test case 1: thick sandwich structure. Geometry and material properties. The plate is simply supported on all edges (SSSS).**

**Figure 15. Test case 2: moderately thick two-layered structure (see References [62] and [63])**

**Figure 16. Test case 3: thick sandwich structure with a very high Face-to-Core Stiffness Ratio (FCSR).**
VI. Results

A. Test Case 1

A fourth-order AHSDT designated as $EEE PVD_{444}$ (see also Figures 4 and 5) is the baseline theory. This baseline theory is modified by enhancing for example one displacement variable with MZZF to take into account the Zig-Zag effects. This is the case of theories $ZEE PVD_{444}$, $EZE PVD_{444}$, and $EEZ PVD_{444}$. Figure 17 shows that the dimensionless in-plane displacements are marginally improved when the displacement in the $z$ direction is enhanced with MZZF. Similar effect is also observed for the in-plane stresses (see Figure 22), transverse shear stresses (see Figure 27), and transverse displacement and normal stress (see Figure 32). If only one in-plane displacement is enhanced with MZZF (theories $ZEE PVD_{444}$ and $EEZ PVD_{444}$) the corresponding displacement is improved whereas this is not the case for the other displacement variables. For example, from Figure 17 it is deduced that enhancing the displacement in the $y$ direction with MZZF (theory $EZE PVD_{444}$) greatly improves the representation of $u_y$ but it does not benefit the quality of the representation for the dimensionless displacement $u_x$. The transverse shear stresses are not greatly improved if just the displacement in the $y$ direction is enhanced with MZZF (see Figure 27). Similar observation could be made for the in-plane stresses (Figure 22) and transverse normal stress (Figure 32). It is also possible to observe that the transverse displacement is improved.

Enhancing just the displacement in the $x$ direction (theory $ZEE PVD_{444}$) has similar numerical performances (see Figures 18, 23, 28, and 33).

Enhancing both the in-plane displacements with MZZF (theory $ZZE PVD_{444}$) provides a great benefit and the results are in excellent agreement with the elasticity solution (see Figures 18, 23, 28, and 33). Enhancing all displacement variables with MZZF (theory $ZZZ PVD_{444}$) provides an even better approximation (see Figures 18, 23, 28, and 33).

A more refined enhancement could be performed by using a Layer Wise description instead of MZZF. This means, for example, to use theory $EEE PVD_{444}$ instead of theory $ZEE PVD_{444}$, or theory $LDE PVD_{444}$ instead of theory $ZZE PVD_{444}$. From Figures 17-36 it is deduced that enhancing only one variable does not produce a significant improvement even if a Layer Wise description is adopted instead of just MZZF. As for the case of Zig Zag enhancement via MZZF, if both the in-plane displacements present a Layer Wise description the results are greatly improved compared to the baseline theory.

Different combinations of orders of expansion and type of discretization (Layer Wise, ESL with and without Zig-Zag enhancement, see for example theory $ZLE PVD_{432}$) are also investigated. Figures 20, 21, 25, 26, 30, 31, 35, and 36 show these cases. It is deduced that the discretization of the in-plane displacement variables...
is crucial for better performance. The transverse displacement is better approximated when a Zig-Zag or Layer Wise type of expansion is adopted for it (provided that the in-plane variables are well described). This is for example deduced by direct inspection of Figure 34.

All the results relative to Test Case 1 were obtained by using an analytical approach (Navier Type solution).

Figure 17. Test case 1: dimensionless displacements $\tilde{u}_x = u_x \frac{[E_{22}]_{\text{Layer 2}}}{x} \frac{h}{h}$ and $\tilde{u}_y = u_y \frac{[E_{22}]_{\text{Layer 2}}}{x} \frac{h}{h}$.

Figure 18. Test case 1: dimensionless displacements $\tilde{u}_x = u_x \frac{[E_{22}]_{\text{Layer 2}}}{x} \frac{h}{h}$ and $\tilde{u}_y = u_y \frac{[E_{22}]_{\text{Layer 2}}}{x} \frac{h}{h}$. 

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Figure 19. Test case 1: dimensionless displacements $\tilde{u}_x = u_x \frac{[E_{22}]_{Layer\ 2}}{\rho^2 h(\frac{x}{a})}$ and $\tilde{u}_y = u_y \frac{[E_{22}]_{Layer\ 2}}{\rho^2 h(\frac{y}{b})}$.

Figure 20. Test case 1: dimensionless displacements $\tilde{u}_x = u_x \frac{[E_{22}]_{Layer\ 2}}{\rho^2 h(\frac{x}{a})}$ and $\tilde{u}_y = u_y \frac{[E_{22}]_{Layer\ 2}}{\rho^2 h(\frac{y}{b})}$.
Figure 21. Test case 1: dimensionless displacements $\tilde{u}_x = u_x \left[ \frac{E_{22}}{\sigma^{*}(\frac{h}{a})^3} \right]_{\text{Layer 2}}$ and $\tilde{u}_y = u_y \left[ \frac{E_{22}}{\sigma^{*}(\frac{h}{a})^3} \right]_{\text{Layer 2}}$.

Figure 22. Test case 1: dimensionless in-plane stresses $\tilde{\sigma}_{xx} = \frac{\sigma_{xx}}{\sigma^{*}(\frac{h}{a})^3}$ and $\tilde{\sigma}_{yy} = \frac{\sigma_{yy}}{\sigma^{*}(\frac{h}{a})^3}$. Classical Form of Hooke’s Law has been used to calculate these stresses.
Figure 23. Test case 1: dimensionless in-plane stress $\tilde{\sigma}_{xx} = \frac{\sigma_{xx}}{\pi^2 \left( \frac{h}{a} \right)^2}$ and $\tilde{\sigma}_{yy} = \frac{\sigma_{yy}}{\pi^2 \left( \frac{h}{a} \right)^2}$. Classical Form of Hooke’s Law has been used to calculate these stresses.

Figure 24. Test case 1: dimensionless in-plane stress $\tilde{\sigma}_{xx} = \frac{\sigma_{xx}}{\pi^2 \left( \frac{h}{a} \right)^2}$ and $\tilde{\sigma}_{yy} = \frac{\sigma_{yy}}{\pi^2 \left( \frac{h}{a} \right)^2}$. Classical Form of Hooke’s Law has been used to calculate these stresses.

B. Test Case 2

Test Case 2 is used to validate the FEM implementation of GUF. Table 1 presents the comparison with some available results obtained from the literature. In particular, several different theories with ESL and LW axiomatic descriptions for the different variables are compared with the analytical solution and other numerical evaluations. In particular these evaluations are obtained with the Higher Order Shear and Normal Deformable Plate Theory (HONSDPT) which accounts for both the transverse normal and the transverse shear deformations and uses Legendre polynomials as basis functions. The in-plane discretization used to obtain the results of Reference [62] was a meshless approach.

C. Test Case 3

Test Case 3 is very challenging because the skins are extremely stiffer than the core. Figures 37 and 38 compare a large amount of theories with some variables described in ESL sense and others described in LW sense. A parabolic LW theory $L_{LL}PVD_{222}$ presents excellent correlation with the elasticity solution. However, all the Equivalent Single Layer models present serious difficulties to converge to the exact solution. For example, even a tenth order fully Zig-Zag theory ($Z_{10}Z_{10}PVD_{101010}$) presents a non-negligible error compared to the elasticity solution (see Figure 38). All the investigations of Test Case 3 were performed by using the Navier type solution.
Figure 25. Test case 1: dimensionless in-plane stress \( \hat{\sigma}_{xx} = \frac{\sigma_{xx}}{z \text{P} \left( \frac{a h}{R} \right)^2} \) and \( \hat{\sigma}_{yy} = \frac{\sigma_{yy}}{z \text{P} \left( \frac{a h}{R} \right)^2} \). Classical Form of Hooke’s Law has been used to calculate these stresses.

Figure 26. Test case 1: dimensionless in-plane stress \( \hat{\sigma}_{xx} = \frac{\sigma_{xx}}{z \text{P} \left( \frac{a h}{R} \right)^2} \) and \( \hat{\sigma}_{yy} = \frac{\sigma_{yy}}{z \text{P} \left( \frac{a h}{R} \right)^2} \). Classical Form of Hooke’s Law has been used to calculate these stresses.

Figure 27. Test case 1: dimensionless transverse shear stresses \( \hat{\sigma}_{xz} = \frac{\sigma_{xz}}{z \text{P} \left( \frac{a h}{R} \right)^2} \) and \( \hat{\sigma}_{zy} = \frac{\sigma_{zy}}{z \text{P} \left( \frac{a h}{R} \right)^2} \). The integration of the equilibrium equations has been used to calculate these stresses.
Figure 28. Test case 1: dimensionless transverse shear stresses $\bar{\sigma}_{xx} = \frac{\sigma_{xx}}{\tau_{xx}(\frac{a}{b})}$ and $\bar{\sigma}_{xy} = \frac{\sigma_{xy}}{\tau_{xy}(\frac{a}{b})}$. The integration of the equilibrium equations has been used to calculate these stresses.

Figure 29. Test case 1: dimensionless transverse shear stresses $\bar{\sigma}_{xx} = \frac{\sigma_{xx}}{\tau_{xx}(\frac{a}{b})}$ and $\bar{\sigma}_{xy} = \frac{\sigma_{xy}}{\tau_{xy}(\frac{a}{b})}$. The integration of the equilibrium equations has been used to calculate these stresses.

Figure 30. Test case 1: dimensionless transverse shear stresses $\bar{\sigma}_{xx} = \frac{\sigma_{xx}}{\tau_{xx}(\frac{a}{b})}$ and $\bar{\sigma}_{xy} = \frac{\sigma_{xy}}{\tau_{xy}(\frac{a}{b})}$. The integration of the equilibrium equations has been used to calculate these stresses.
Figure 31. Test case 1: dimensionless transverse shear stresses \( \bar{\sigma}_{zx} = \frac{\sigma_{zx}}{\frac{1}{2}Eh^3} \) and \( \bar{\sigma}_{zy} = \frac{\sigma_{zy}}{\frac{1}{2}Eh^3} \). The integration of the equilibrium equations has been used to calculate these stresses.

Figure 32. Test case 1: dimensionless displacement \( \bar{u}_z = u_z \frac{100|Ez|^{3/2}h}{Eh^2} \) and transverse normal stress \( \bar{\sigma}_{zz} = \frac{2\sigma_{zz}}{Eh^2} \).

Figure 33. Test case 1: dimensionless displacement \( \bar{u}_z = u_z \frac{100|Ez|^{3/2}h}{Eh^2} \) and transverse normal stress \( \bar{\sigma}_{zz} = \frac{2\sigma_{zz}}{Eh^2} \).
Figure 34. Test case 1: dimensionless displacement $\hat{u}_z = u_z \frac{E_{22}}{E_{12}h} \eta^2$ and transverse normal stress $\hat{\sigma}_{zz} = \frac{\sigma_{zz}}{\sigma_{pt}}$.

Figure 35. Test case 1: dimensionless displacement $\hat{u}_z = u_z \frac{E_{22}}{E_{12}h} \eta^2$ and transverse normal stress $\hat{\sigma}_{zz} = \frac{\sigma_{zz}}{\sigma_{pt}}$.

Figure 36. Test case 1: dimensionless displacement $\hat{u}_z = u_z \frac{E_{22}}{E_{12}h} \eta^2$ and transverse normal stress $\hat{\sigma}_{zz} = \frac{\sigma_{zz}}{\sigma_{pt}}$. 
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<th>$\bar{\sigma}_{yy}(\frac{a}{2}, \frac{b}{2}, +\frac{b}{2})$</th>
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<td>0.3840</td>
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Present FEM Evaluations

| EE E PV D_{333} | 0.637                                 | -0.4657                                        | 0.3847                                          | 0.0683                                   |
| EE Z PV D_{333} | 0.637                                 | -0.4655                                        | 0.3846                                          | 0.0684                                   |
| EE E PV D_{333} | 0.637                                 | -0.4653                                        | 0.3844                                          | 0.0684                                   |
| EE E PV D_{333} | 0.637                                 | -0.4659                                        | 0.3845                                          | 0.0682                                   |
| EE E PV D_{333} | 0.638                                 | -0.4666                                        | 0.3849                                          | 0.0684                                   |
| EE E PV D_{333} | 0.637                                 | -0.4655                                        | 0.3849                                          | 0.0683                                   |
| EE E PV D_{333} | 0.647                                 | -0.4712                                        | 0.3906                                          | 0.0694                                   |
| EE E PV D_{333} | 0.637                                 | -0.4658                                        | 0.3847                                          | 0.0682                                   |
| EE E PV D_{333} | 0.637                                 | -0.4656                                        | 0.3845                                          | 0.0683                                   |
| EE E PV D_{333} | 0.637                                 | -0.4654                                        | 0.3844                                          | 0.0683                                   |
| EE E PV D_{333} | 0.648                                 | -0.4721                                        | 0.3907                                          | 0.0695                                   |
| EE E PV D_{333} | 0.648                                 | -0.4720                                        | 0.3906                                          | 0.0696                                   |
| EE E PV D_{333} | 0.648                                 | -0.4718                                        | 0.3905                                          | 0.0696                                   |

Table 1. Test case 2 (see Reference [62]). Finite element evaluations and comparison with the literature. In the present evaluations a $6 \times 6$ mesh was used to simulate a quarter of the plate (the symmetry conditions were imposed). $\bar{\sigma}_{zy}$ is calculated by using the integration of the equilibrium equations.
Figure 37. Test case 3: dimensionless displacement $\hat{u}_x = u_x \frac{E_{core}}{s_F h(\frac{z}{R})^2}$

Figure 38. Test case 3: dimensionless displacement $\hat{u}_x = u_x \frac{E_{core}}{s_F h(\frac{z}{R})^2}$
VII. Conclusions

This work presented a finite element implementation of the Generalized Unified Formulation for the first time. Moreover, GUF was further generalized: it is now possible to use any type of discretization (Equivalent Single Layer or Layer Wise) for all the displacement variables. Partially Layer Wise Advanced Zig-Zag and Higher Order Shear Deformation Theories (PLAZHSDTs) were discussed and compared with Partially Layer-Wise Advanced Higher Order Shear Deformation Theories (PLAHSDTs), Partially Zig-Zag Advanced Higher Order Shear Deformation Theories (PZZAHSDTs), Advanced Zig-Zag Theories (AZZTs), and Advanced Higher Order Shear Deformation Theories (AHSDTs). Different orders of expansions where also explored within the GUF multi-theory architecture framework. With GUF the selection of the orders of expansion can be exactly performed without the adoption of the penalty method which could provide numerical ill conditioning of the stiffness matrices. This very large amount of new theories was obtained from just six theory-invariant $1 \times 1$ arrays (the kernels) by expansion of 4 indices. The expansion with respect to the first two indices is used to generate the matrix at layer and nodal levels. A subsequent expansion with respect to the second pair of indices is used to generate the stiffness matrix at element and layer levels. Then, a special assembling procedure in the thickness direction takes care of the actual type of axiomatic description for the different variables and the compatibility of the displacements is enforced at that stage. This work focused the attention on displacement-based formulations (i.e., the primary unknowns are the displacements and the variational statement is the Principle of Virtual Displacements) but it can be easily extended to other variational statements which could have some stresses as primary unknowns. In that case different $1 \times 1$ kernels need to be defined but the procedure is very similar.

With this new extension the range of types of theories that can be generated with the GUF multi-theory architecture is very large and can cover the classical models, the advanced formulations and all the intermediate approaches. This makes the GUF architecture particularly indicated in probabilistic and optimization problems.

Based on the investigations presented in this work, the following facts could be observed:

- If the in-plane displacement models are enhanced with the adoption of Murakami’s Zig-Zag Function (MZZF) the overall response is greatly improved and the axiomatic model is quite competitive as far as the computational cost and accuracy of the results are concerned.

- If only one in-plane displacement model is enhanced with the adoption MZZF the results are slightly improved.

- If only the transverse displacement is enhanced with the adoption MZZF the results are only marginally improved.

- If a Layer Wise description is used for a single variable only, the numerical performance is not improved significantly.

- If a Layer Wise description is used for both the in-plane displacements the results are in excellent correlation with the elasticity solution and are better than the ones obtained with a Zig-Zag enhancement of the same variables. However, the computational cost is higher.

- For very challenging cases in which the three dimensional effects are significant a Layer-Wise approach is practically mandatory for a best use of the computational resources. It was demonstrated that even very high order Zig-Zag theories cannot provide sufficient accuracy (Test Case 3). GUF can generate theories with a desired level of accuracy and it is a matter only of a convergence test to find the best axiomatic model which is more suitable for the problem under investigation.

Future extensions of this work will include mixed formulations and other advanced displacement-based formulations such as the ones presented in Reference [65].
References


