An Invariant Model for any Composite Plate Theory and FEM applications: the Generalized Unified Formulation

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The Generalized Unified Formulation (GUF) presented here is a modern approach for the Finite Element analysis of sandwich plates and in general multilayered structures. GUF is a comprehensive formulation which includes practically all possible axiomatic theories. Any type of theory with any combination of orders of expansion for the different displacement variables can be obtained from the expansion of six \(1 \times 1\) arrays (the kernels or fundamental nuclei of the Generalized Unified Formulation). Each of the displacement variables is independently treated and different orders of expansions for the different unknowns can be chosen. Since infinite combinations can be freely chosen for the displacements, GUF allows the user to write, with a single invariant formulation implemented in a single FEM code, \(\infty^3\) Higher-order Shear Deformation Theories (HSDT), \(\infty^3\) Zig-Zag Theories (ZZT) and \(\infty^3\) Layer-Wise Theories (LWT). The six independent fundamental nuclei are formally invariant with respect to the order used for the expansion or with respect to the type of theory. GUF is a powerful versatile tool: the user can freely decide the type of theory (e.g., LWT or ZZT) and the orders used in the expansions. Thus, a code based on GUF can be fast (less terms can be used in the expansions of the unknwons) when the case under investigation is less demanding or when a large number of runs is required (optimization problems or probabilistic studies) and the software can be very accurate (layerwise models with high orders used for the unknowns) when the problem is very challenging and the accuracy is more important than the CPU time.

GUF can be implemented in existing commercial codes and “intelligent” software can be developed. The codes can, in fact, use any theory or order of expansion to attempt the convergence in case of complex three-dimensional problems with localized effects without the actual need of solid elements. This is another advantage because the number of Degrees of Freedom could be reduced without loosing in computational efficiency and with the advantage of having the two dimensional mesh typical of plate and shell models. The Generalized Unified Formulation can be also adopted for mixed variational statements. In such cases the number of independent kernels would be different but their size would still be \(1 \times 1\) arrays. Multifield problems such as thermoelastic applications and multilayered plates embedding piezo-layers can also be analyzed as well.

This paper assesses bending of sandwich structures. Analytical and elasticity solution are compared. The effect of the Zig-Zag form of the displacement is discussed.

Nomenclature

\begin{align*}
  u_x & \quad \text{Displacement in the } x \text{ direction} \\
  u_y & \quad \text{Displacement in the } y \text{ direction} \\
  u_z & \quad \text{Displacement in the } z \text{ direction} \\
  \hat{u}_x, \hat{u}_y & \quad \text{Dimensionless in-plane displacements} \\
  \hat{u}_z & \quad \text{Dimensionless out-of-plane displacement} \\
  \hat{\sigma}_{xx}, \hat{\sigma}_{yy}, \hat{\sigma}_{zz} & \quad \text{Dimensionless out-of-plane stresses} \\
  \hat{\sigma}_{zx}, \hat{\sigma}_{zy}, \hat{\sigma}_{xy} & \quad \text{Dimensionless in-plane stresses}
\end{align*}

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$x, y$ In-plane coordinates
$z$ Out-of-plane coordinate
$z_{bot}$ $z$ coordinate of the bottom surface of layer $k$
$z_{top}$ $z$ coordinate of the top surface of layer $k$
$m, n$ Wave numbers
$\varepsilon_{xx}, \varepsilon_{yy}, \gamma_{xy}$ In-plane strains
$\varepsilon_{zz}, \gamma_{xz}, \gamma_{yz}$ Out-of-plane strains
$\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ In-plane stresses
$\sigma_{zz}, \sigma_{xz}, \sigma_{yz}$ Out-of-plane stresses
$x^F_1, y^F_1, z^F_1$ Known functions used in the expansion along the thickness
$x^F_b, y^F_b, z^F_b$ Known functions used in the expansion along the thickness
$x^F_k, y^F_k, z^F_k$ Known functions used in the expansion along the thickness
$\alpha$ Thickness primary master index
$\beta$ Thickness secondary master index
$N_{ux}$ Order of expansion used for the displacement $u_x$
$N_{uy}$ Order of expansion used for the displacement $u_y$
$N_{uz}$ Order of expansion used for the displacement $u_z$
$\zeta_k$ Non-dimensional coordinate (see equation 39)
$K_{ux, \alpha, \beta, u_x}$ Invariant Kernel (1 × 1 array)
$K_{ux, \alpha, \beta, u_y}$ Invariant Kernel (1 × 1 array)
$K_{ux, \alpha, \beta, u_z}$ Invariant Kernel (1 × 1 array)
$K_{uy, \alpha, \beta, u_y}$ Invariant Kernel (1 × 1 array)
$K_{uy, \alpha, \beta, u_z}$ Invariant Kernel (1 × 1 array)
$K_{uz, \alpha, \beta, u_z}$ Invariant Kernel (1 × 1 array)
$t^{D}_{ux, \alpha, \beta, u_x}$ Invariant pressure Kernel (1 × 1 array)
$t^{D}_{ux, \alpha, \beta, u_y}$ Invariant pressure Kernel (1 × 1 array)
$t^{D}_{ux, \alpha, \beta, u_z}$ Invariant pressure Kernel (1 × 1 array)
$b^{D}_{ux, \alpha, \beta, u_x}$ Invariant pressure Kernel (1 × 1 array)
$b^{D}_{ux, \alpha, \beta, u_y}$ Invariant pressure Kernel (1 × 1 array)
$b^{D}_{ux, \alpha, \beta, u_z}$ Invariant pressure Kernel (1 × 1 array)
$\varepsilon_p$ Vector containing the in-plane strains
$\varepsilon_p$ Vector containing the in-plane stresses
$\varepsilon_n$ Vector containing the out-of-plane strains
$\varepsilon_n$ Vector containing the out-of-plane stresses

Subscript
$k$ Referred to layer $k$
$u_x$ Referred to displacement $u_x$
$u_y$ Referred to displacement $u_y$
$u_z$ Referred to displacement $u_z$
$G$ Quantity calculated by using the geometric relations (see equation 11)
$H$ Quantity calculated by using Hooke’s law (see equation 92)

Superscript
$T$ Transpose
$k$ Referred to layer $k$
I. Introduction

A. Background and Motivation

The Finite Element Method \(1-3\) is a powerful tool widely used in the industry. With the growing computational capabilities of the modern computers increasingly complex problems can be solved. Most of the aerospace structures can be modeled with shell and plate models. Therefore, the importance of finite element applications of such structures is great. Different problems have been solved in the past: free vibration of plates, \(4-9\) analysis of sandwich structures, \(10-13\) thermoelastic problems, \(14\) piezoelectric plates \(15\) and multifield loadings. \(16\). A very large amount of literature of theory of plates is available and it will not be discussed here. With focus on axiomatic methods, each existing approach presents a range of applicability and when the hypotheses used to formulate the theory are no longer satisfied the approach has to be replaced with another one usually named by the authors as “refined theory” or “improved theory”. In the framework of the mechanical case the Classical Plate Theory (CPT), also known as Kirchoff theory, \(17\) has the advantage of being simple and reliable for thin plates. However, if there is strong anisotropy of the mechanical properties, or if the composite plate is relatively thick, other advanced models such as First-order Shear Deformation Theory (FSDT) are required. \(18-20\) Higher-order Shear Deformation Theories (HSDT) have also been used, \(5-7\) giving the possibility to increase the accuracy of numerical evaluations for moderately thick plates. But even these theories are not sufficient if local effects are important or accuracy in the calculation of transverse stresses is sought. Therefore, more advanced plate theories have been developed to include zig-zag effects. \(21-32\) When a new problem with unknown behavior is analyzed, the user may attack it with the most simple formulation or with the most advanced and refined models or with an intermediate solution which is a compromise between the accuracy and the CPU time needs. This operation is usually costly. For example, the different capabilities required for this approach are not available. In that case in-house software is often created. But this in-house software obviously does not contain all possible models between the least, and usually fast, accurate one and the most advanced, and usually time consuming, one. Even if different approaches are implemented, each one requires a different formulation of the FE stiffness matrix and the creation of an independent software. A major problem may arise in that case. The implemented capability may be not sufficient to solve the challenging case the user is facing or the CPU time is unacceptable and a less accurate theory would be sufficient. These aspects are also problem-dependent and even if the solution of a previous case was satisfactory, the same software may be no longer adequate for a new task. Therefore, an extra cost must be taken into account to face a new situation.

The problem of adequately modeling the behavior of the real structure is then very challenging. For example, if two-dimensional multilayered structures cases are considered, it may be necessary to abandon the so called Equivalent Single Layer Models (ESLM) and adopt Layerwise Models (LWM), \(8,33-40\) in which the variables are layer-dependent. If the anisotropy in the thickness direction is very strong, the accuracy can be preserved by increasing the number of layers used in the mathematical model or/and by increasing the orders of the axiomatic expansion along the thickness for the different variables. If this approach is chosen, the use of solid elements can be avoided and the advantage of having a two-dimensional mesh can be preserved. Versatility is also required when a commercial or in-house software is considered. Unfortunately this is not usually the case. An optimization problem or a probabilistic analysis (Monte Carlo) needs to be very fast whereas a detailed study of the interlaminar stresses requires an advanced model.

What is proposed in this paper meets all the above mentioned requirements and is a powerful tool for the modelization of most structures in the aerospace field.

B. What are the contributions of this work

An invariant model, named Generalized Unified Formulation (GUF) is introduced. It can be considered a generalization of the Unified Formulation (GUF) \(41\) (named here as Carrera’s Unified Formulation). With the present GUF all the needs earlier mentioned are satisfied. In particular, the Generalized Unified Formulation allows to treat each displacement variable independently from the others and this could lead the derivation of new FEM numerical techniques. The orders of each variable are freely chosen by the user and can be different than the orders used for the thickness expansion of the other variables. This property means that practically
all “classical” Equivalent Single Layer Models can be addressed by the Generalized Unified Formulation. For example, an advanced Higher-order Shear Deformation Theory in which the in-plane displacements are expanded along the thickness by using a cubic polynomial and in which the out-of-plane displacement is parabolic can be represented with GUF as well as the theory in which the orders for the in-plane displacements are parabolic and the out-of-plane displacement is linear. Any combination is allowed. But what is important is that all the possible combinations that can be generated (which are $\infty^3$ since three displacement variables $u_x$, $u_y$, and $u_z$ are here considered) are obtained from the expansion of six kernels which are invariant with respect to the order adopted for the displacements variables. That is, all the Higher-order Shear Deformation Theories with any combination of orders can be generated from the same six invariant kernels. The kernels of the Generalized Unified Formulation are $1 \times 1$ matrices. If other variational statements different than the Principle of Virtual Displacements are used, the number of independent kernels and their mathematical form will be different but the size will always be $1 \times 1$.

It is possible to demonstrate that when the so called Zig-Zag form of the displacements is included by adopting Murakami’s Zig-Zag Function,$^{21}$ the same invariant kernels can describe the $\infty^3$ Zig-Zag theories that could be generated by changing the orders of the expansions of the displacements. But these features are not limited to the Equivalent Single Layer Models. All possible Layerwise Theories with any combination of orders can be obtained, again, from the same six invariant $1 \times 1$ kernels. The GUF clearly allows to have all these infinite models in the same software. From the least accurate to the most advanced one. The freedom to change the orders of expansion in a layerwise model allows to reach a quasi-3D solution without the actual need of solid elements and so with the preservation of the advantages of having a two-dimensional mesh. This is very useful when local effects are important or when the analyzed structure may not be considered as a “thin structure” and so when the classical two-dimensional formulation fails.

Another advantage of the GUF is in the possibility of a numerical “experiment” or a “sensitivity” analysis of the problem under investigation. For example, it is possible to explore if a particular configuration of the constraints and loads is affected more by the order of expansion of one variable with respect to the orders used for the other variables. This feature is particularly appealing in the mixed or multifield cases in which the order of expansion of some unknown quantities may be determinant to have the correct numerical simulation. But this is not all. The Generalized Unified Formulation can be used to “adapt” the software to the optimization or probabilistic simulation. The user can in fact study with a small number of runs the best combination of type of theory (e.g. Equivalent Single Layer Theory with Zig-Zag Function vs a Higher-order Shear Deformation Theory) and orders that minimize the CPU time and maximize the accuracy within the requested level. Then the user can start the optimization or probabilistic study knowing that the software gives the best performances because it is “tailored” to the case under investigation. The Generalized Unified Formulation could also be used to design automatic and “intelligent” FEM codes: the codes could understand (with a designed algorithm) where the accuracy is required the most and so they could “try” any of the types of theories and any of the $\infty^3$ combinations allowed by GUF and improve the accuracy of the prediction. As stated earlier this capability is not limited to the pure mechanical case presented in this paper and it can be easily extended to the multifield problems and functionally graded materials.

A final but not less important advantage of the Generalized Unified Formulation is in the educational possibilities for young engineers. The engineers could learn a single formulation (GUF) and apply all the powerful features of GUF to a very vast class of problems without the need of learning new formulations of commercial softwares. With the Generalized Unified Formulation the “classical” axiomatic approaches and the distinction between theories are just an exercise and the user can move from one approach to another one without difficulties. The Generalized Unified Formulation can be then considered a modern view of multilayered plates and shells with possible applications in high fidelity codes in the different disciplines such as aeroelasticity.

II. Classification of the theories

The main feature of the Generalized Unified Formulation is that the descriptions of Layerwise Theories, Higher-order Shear Deformation Theories and Zig-Zag Theories of any combination of orders do not show any formal differences and can all be obtained from six invariant kernels. So, with just one theoretical model an infinite number of different approaches can be considered. For example, in the case of moderately thick plates a higher order theory could be sufficient but for thick plates layerwise models may be required. With GUF the two approaches are formally identical because the kernels are invariant with respect to the type of theory.
In the present work the concepts of type of theory and class of theories are introduced. The following types of displacement-based theories are discussed. The first type is named as Advanced Higher-order Shear Deformation Theories (AHSDT). These theories are Equivalent Single Layer models because the displacement field is unique and independent of the number of layers. The effects of the transverse normal strain $\varepsilon_{zz}$ are retained.

The second type of theories is named as Advanced Higher-order Shear Deformation Theories with Zig-Zag effects included (AHSDTZ). These theories are Equivalent Single Layer models and the so called Zig-Zag form of the displacements is taken into account by using Murakami’s Zig-Zag Function (MZZF). The effects of the transverse normal strain $\varepsilon_{zz}$ are included. The third type of theories is named Advanced LayerWise Theories (ALWT). These theories are the most accurate ones because all the displacements have a layerwise description. The effects of the transverse normal strain $\varepsilon_{zz}$ are included as well. These models are necessary when local effects need to be described. The price is of course (in FEM applications) in higher computational time. An infinite number of theories which have a particular logic in the selection of the used orders of expansion is defined as class of theories. For example, the infinite layerwise theories which have the displacements $u_x$, $u_y$ and $u_z$ expanded along the thickness with a polynomial of order $N$ are a class of theories. The infinite theories which have the in-plane displacements $u_x$ and $u_y$ expanded along the thickness with order $N$, the out of plane displacement expanded along the thickness with order $N$, the out of plane displacement expanded along the thickness with order $N$ are another class of theories.

III. The Generalized Unified Formulation for Multilayered Composite Plates

Both layerwise and Equivalent Single Layer models are axiomatic approaches. That is, the unknowns are expanded along the thickness by using a chosen series of functions. When the Principal of Virtual Displacements is used, the unknowns are the displacements $u_x$, $u_y$ and $u_z$. When other variational statements are used the unknowns may also be all or some of the stresses and other quantities as well (multifield case).

The Generalized Unified Formulation is introduced here considering a generic layer $k$ of a multilayered plate structure. This is the most general approach and the Equivalent Single Layer theories, which consider the displacement unknowns to be layer-independent, can be derived from this formulation with some simple formal techniques as will be demonstrated in this paper. Consider a theory denoted as Theory I, in which the displacement in $x$ direction $u_k^x$ has four Degrees of Freedom. Here by Degrees of Freedom it is intended the number of unknown quantities that are used to expand a variable. In the case under examination four Degrees of Freedom for the displacement $u_k^x$ means that four unknowns are considered. Each unknown multiplies a known function of the thickness coordinate $z$. Where the origin of the coordinate $z$ is measured is not important. However, from a practical point of view it is convenient to assume that the middle plane of the plate is also the plane with $z = 0$. This assumption does not imply that there is a symmetry with respect to the plane $z = 0$. The formulation is general.

![Multilayered plate: notations and definitions.](image)

Figure 1. Multilayered plate: notations and definitions.

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For layer $k$ the following relation holds: $z_{\text{bot}_k} \leq z \leq z_{\text{top}_k}$. $z_{\text{bot}_k}$ is the global coordinate $z$ of the bottom surface of layer $k$ and $z_{\text{top}_k}$ is the global coordinate $z$ of the top surface of layer $k$ (see Figure 1). $h_k = z_{\text{top}_k} - z_{\text{bot}_k}$ is the thickness of layer $k$ and $h$ is the thickness of the plate.

In the case of Theory I, $u^k$ is expressed as follows:

$$
\begin{align*}
\mathbf{u}^k (x, y, z) &= \begin{bmatrix} \text{known} & \text{unknown#1} & \text{known} & \text{unknown#2} \\
\text{known} & \text{unknown#3} & \text{known} & \text{unknown#4} \\
\end{bmatrix} \\
&= \mathbf{f}^k_1 (z) \cdot \mathbf{u}_{x_1}^k (x, y) + \mathbf{f}^k_2 (z) \cdot \mathbf{u}_{x_2}^k (x, y) + \mathbf{f}^k_3 (z) \cdot \mathbf{u}_{x_3}^k (x, y) + \mathbf{f}^k_4 (z) \cdot \mathbf{u}_{x_4}^k (x, y) \\
&\quad \text{z}_{\text{bot}_k} \leq z \leq \text{z}_{\text{top}_k} \quad (1)
\end{align*}
$$

The functions $\mathbf{f}^k_1 (z)$, $\mathbf{f}^k_2 (z)$, $\mathbf{f}^k_3 (z)$ and $\mathbf{f}^k_4 (z)$ are known functions (axiomatic approach). These functions could be, for example, a series of trigonometric functions of the thickness coordinate $z$. Polynomials (or even better orthogonal polynomials) could be selected. In the most general case each layer has different functions. For example, $\mathbf{f}^k_1 (z) \neq \mathbf{f}^{k+1}_1 (z)$. The next formal step is to modify the notation.

The following functions are defined:

$$
\begin{align*}
\mathbf{F}^k_1 (z) &= \mathbf{f}^k_1 (z) \\
\mathbf{F}^k_2 (z) &= \mathbf{f}^k_2 (z) \\
\mathbf{F}^k_3 (z) &= \mathbf{f}^k_3 (z) \\
\mathbf{F}^k_4 (z) &= \mathbf{f}^k_4 (z) \\
\end{align*}
$$

(2)

The logic behind these definitions is the following. The first function $\mathbf{f}^k_1 (z)$ is defined as $\mathbf{F}^k_1$. Notice the superscript $x$. It was added to clarify that the displacement in $x$ direction, $u^k_x$, is under investigation. The subscript $t$ identifies the quantities at the “top” of the plate and, therefore, will be useful in the assembling of the stiffness matrices in the thickness direction. This aspect will be discussed later.

The last function $\mathbf{f}^k_4 (z)$ is defined as $\mathbf{F}^k_4$. Notice again the superscript $x$. The subscript $b$ means “bottom” and, again, its utility will be clear when the matrices are assembled.

The intermediate functions $\mathbf{f}^k_2 (z)$ and $\mathbf{f}^k_3 (z)$ are defined simply as $\mathbf{F}^k_2$ and $\mathbf{F}^k_3$. To be consistent with the definitions of equation (2), the following unknown quantities are defined:

$$
\begin{align*}
\mathbf{u}_{x_1}^k (x, y) &= \mathbf{u}_{x_1} (x, y) \\
\mathbf{u}_{x_2}^k (x, y) &= \mathbf{u}_{x_2} (x, y) \\
\end{align*}
$$

(3)

Using the definitions reported in equations (2) and (3), equation (1) can be rewritten as

$$
\begin{align*}
\mathbf{u}^k (x, y, z) &= \begin{bmatrix} \text{known} & \text{unknown#1} & \text{known} & \text{unknown#2} \\
\text{known} & \text{unknown#3} & \text{known} & \text{unknown#4} \\
\end{bmatrix} \\
&= \mathbf{\alpha}^k_1 (z) \cdot \mathbf{u}_{x_1}^k (x, y) + \mathbf{\alpha}^k_2 (z) \cdot \mathbf{u}_{x_2}^k (x, y) + \mathbf{\alpha}^k_3 (z) \cdot \mathbf{u}_{x_3}^k (x, y) + \mathbf{\alpha}^k_4 (z) \cdot \mathbf{u}_{x_4}^k (x, y) \\
&\quad \text{z}_{\text{bot}_k} \leq z \leq \text{z}_{\text{top}_k} \quad (4)
\end{align*}
$$

It is supposed that each function of $z$ is a polynomial. The order of the expansion is then 3 and indicated as $N^k_{\text{top}}$. Each layer has in general a different order. Thus, in general $N^k_{\text{top}} \neq N^{k+1}_{\text{top}}$. If the functions of $z$ are not polynomials (for example, this is the case if trigonometric functions are used) then $N^k_{\text{top}}$ is just a parameter related to the number of terms or Degrees of Freedom used to describe the displacement $u^k_x$ in the thickness direction. This concept will be clear later. The expression representing the displacement $u^k_x$ (see equation (4)) can be put in a compact form typical of the Generalized Unified Formulation presented here. In particular it is possible to write:

$$
\begin{align*}
\mathbf{u}^k (x, y, z) &= \mathbf{\alpha}^k (z) \cdot \mathbf{u}_{\alpha x}^k (x, y) \\
&\quad \text{z}_{\text{bot}_k} \leq z \leq \text{z}_{\text{top}_k} \\
\end{align*}
$$

(5)

where, in the example, $N^k_{\text{top}} = 3$. The thickness primary master index $\alpha$ has the subscript $u_x$. This subscript from now on will be called slave index. It is introduced to show that the displacement $u_x$ is considered. Figure 2 explains these definitions. Consider another example. Suppose that the displacement $u^k_x$ of a particular theory is expressed with 3 Degrees of Freedom. In that case it is possible to write:

$$
\begin{align*}
\mathbf{u}^k (x, y, z) &= \begin{bmatrix} \text{known} & \text{unknown#1} & \text{known} & \text{unknown#2} & \text{known} & \text{unknown#3} \\
\end{bmatrix} \\
&= \mathbf{f}^k_1 (z) \cdot \mathbf{u}_{x_1}^k (x, y) + \mathbf{f}^k_2 (z) \cdot \mathbf{u}_{x_2}^k (x, y) + \mathbf{f}^k_3 (z) \cdot \mathbf{u}_{x_3}^k (x, y) \\
\end{align*}
$$

(6)
is concluded that equation (1) can be rewritten in the form shown in equation (5) with $N_{u_x}^k = 2$. In general $N_{u_x}^k$ is $DOF_{u_x}^k - 1$, where $DOF_{u_x}^k$ is the number of Degrees of Freedom (at layer level) used for the displacement $u_x^k$. In the case of Zig-Zag theories it is possible to demonstrate that $N_{u_x}^k = DOF_{u_x}^k - 2$ because one Degree of Freedom is used for the Zig-Zag function. The minimum number of Degrees of Freedom is chosen to be 2. This is a choice used to facilitate the assembling in the thickness direction. In fact, the “top” and “bottom” terms will be always present. In the case in which $DOF_{u_x}^k = 2$ the Generalized Unified Formulation is simply

$$u_x^k (x, y, z) = \sum_{\alpha=1}^{\text{unknown#1}} \sum_{\beta=1}^{\text{unknown#2}} \sum_{\gamma=1}^{\text{unknown#3}} \alpha u_x^k (z) \cdot u_{x u_x}^k (x, y) \quad \alpha_{u_x} = t, b$$

(8)

In this particular case the “l” term of equation (5) is not present.

An infinite number of theories can be included in equation (5). It is in fact sufficient to change the value of $N_{u_x}^k$. It should be observed that formally there is no difference between two distinct theories (obtained by changing $N_{u_x}^k$). It is deduced that $\infty^3$ theories can be represented by equation (5).

The other displacements $u_y^k$ and $u_z^k$ can be treated in a similar fashion. The Generalized Unified Formulation for all the displacements is the following:

$$u_y^k = \sum_{\alpha=1}^{\text{unknown#1}} \sum_{\beta=1}^{\text{unknown#2}} \sum_{\gamma=1}^{\text{unknown#3}} \alpha u_y^k (z) \cdot u_{y u_y}^k (x, y) \quad \alpha_{u_y} = t, m, b$$

$$u_z^k = \sum_{\alpha=1}^{\text{unknown#1}} \sum_{\beta=1}^{\text{unknown#2}} \sum_{\gamma=1}^{\text{unknown#3}} \alpha u_z^k (z) \cdot u_{z u_z}^k (x, z) \quad \alpha_{u_z} = t, n, b$$

(9)

In equation (9) for simplicity it is assumed that the type of functions is the same for each layer and that the same number of terms is used for each layer. This assumption will make it possible to adopt the same Generalized Unified Formulation for all types of theories, and layerwise and equivalent single layer theories will not show formal differences. This concept means, for example, that if displacement $u_y$ is approximated with five terms in a particular layer $k$ then it will be approximated with five terms in all layers of the multilayered structure.

Each displacement variable can be expanded in $\infty^3$ combinations. In fact, it is sufficient to change the number of terms used for each variable. Since there are three variables (the displacements $u_x$, $u_y$, and $u_z$), it is concluded that equation (9) includes $\infty^3$ different theories. For now the quantities are defined in a layerwise sense but it will be shown that the same concept is valid for the Equivalent Single Layer cases too.

By adopting the definitions earlier used for the case of 4 Degrees of Freedom it is possible to rewrite equation (6) in the following equivalent form:

$$u_x^k (x, y, z) = \sum_{\alpha=1}^{\text{known}} \sum_{\beta=1}^{\text{unknown#1}} \sum_{\gamma=1}^{\text{unknown#2}} \sum_{\delta=1}^{\text{unknown#3}} \alpha u_x^k (z) \cdot u_{x u_x}^k (x, y)$$

(7)

which can be put again in the form shown in equation (5) with $N_{u_x}^k = 2$. In general $N_{u_x}^k$ is $DOF_{u_x}^k - 1$, where $DOF_{u_x}^k$ is the number of Degrees of Freedom (at layer level) used for the displacement $u_x^k$. In the case of Zig-Zag theories it is possible to demonstrate that $N_{u_x}^k = DOF_{u_x}^k - 2$ because one Degree of Freedom is used for the Zig-Zag function.

In equation (7) there is no term $t, n, b$ which can be put again in the form shown in equation (5) with $N_{u_x}^k = 2$. In general $N_{u_x}^k$ is $DOF_{u_x}^k - 1$, where $DOF_{u_x}^k$ is the number of Degrees of Freedom (at layer level) used for the displacement $u_x^k$. In the case of Zig-Zag theories it is possible to demonstrate that $N_{u_x}^k = DOF_{u_x}^k - 2$ because one Degree of Freedom is used for the Zig-Zag function.

The minimum number of Degrees of Freedom is chosen to be 2. This is a choice used to facilitate the assembling in the thickness direction. In fact, the “top” and “bottom” terms will be always present. In the case in which $DOF_{u_x}^k = 2$ the Generalized Unified Formulation is simply

$$u_x^k (x, y, z) = \sum_{\alpha=1}^{\text{known}} \sum_{\beta=1}^{\text{unknown#1}} \sum_{\gamma=1}^{\text{unknown#2}} \sum_{\delta=1}^{\text{unknown#3}} \alpha u_x^k (z) \cdot u_{x u_x}^k (x, y)$$

(8)

In this particular case the “l” term of equation (5) is not present.

An infinite number of theories can be included in equation (5). It is in fact sufficient to change the value of $N_{u_x}^k$. It should be observed that formally there is no difference between two distinct theories (obtained by changing $N_{u_x}^k$). It is deduced that $\infty^3$ theories can be represented by equation (5).

The other displacements $u_y^k$ and $u_z^k$ can be treated in a similar fashion. The Generalized Unified Formulation for all the displacements is the following:

$$u_y^k = \sum_{\alpha=1}^{\text{known}} \sum_{\beta=1}^{\text{unknown#1}} \sum_{\gamma=1}^{\text{unknown#2}} \alpha u_y^k (z) \cdot u_{y u_y}^k (x, y)$$

$$u_z^k = \sum_{\alpha=1}^{\text{known}} \sum_{\beta=1}^{\text{unknown#1}} \sum_{\gamma=1}^{\text{unknown#2}} \alpha u_z^k (z) \cdot u_{z u_z}^k (x, z)$$

(9)

In equation (9) for simplicity it is assumed that the type of functions is the same for each layer and that the same number of terms is used for each layer. This assumption will make it possible to adopt the same Generalized Unified Formulation for all types of theories, and layerwise and equivalent single layer theories will not show formal differences. This concept means, for example, that if displacement $u_y$ is approximated with five terms in a particular layer $k$ then it will be approximated with five terms in all layers of the multilayered structure.

Each displacement variable can be expanded in $\infty^3$ combinations. In fact, it is sufficient to change the number of terms used for each variable. Since there are three variables (the displacements $u_x$, $u_y$, and $u_z$), it is concluded that equation (9) includes $\infty^3$ different theories. For now the quantities are defined in a layerwise sense but it will be shown that the same concept is valid for the Equivalent Single Layer cases too.
A. Governing Equations

A multilayered structure composed of \( N_l \) layers is considered. The Principle of Virtual Displacements for the case of two pressures applied at the top and bottom of each layer \( k \) is:

\[
\int \int_{\Omega^k}^{z_{\text{top}_k}} [\delta \varepsilon \frac{k T}{p G} \sigma_{p H}^k + \delta \varepsilon \frac{k T}{n e} \sigma_{n H}^k] \, dz \, dx = \delta L_c^k =
\]

\[
+ \int \int_{\Omega^k}^{z_{\text{bot}_k}} [\delta u^k_x (x, y, z = z_{\text{top}_k}) P_{x z}^k (x, y, z) \, dz \, dy +
\]

\[
+ \int \int_{\Omega^k}^{z_{\text{bot}_k}} [\delta u^k_y (x, y, z = z_{\text{top}_k}) P_{y z}^k (x, y, z) \, dz \, dy +
\]

\[
+ \int \int_{\Omega^k}^{z_{\text{bot}_k}} [\delta u^k_z (x, y, z = z_{\text{top}_k}) P_{z z}^k (x, y, z) \, dz \, dy +
\]

\[
+ \int \int_{\Omega^k}^{z_{\text{bot}_k}} [\delta u^k_x (x, y, z = z_{\text{bot}_k}) P_{x z}^k (x, y, z) \, dz \, dy +
\]

\[
+ \int \int_{\Omega^k}^{z_{\text{bot}_k}} [\delta u^k_y (x, y, z = z_{\text{bot}_k}) P_{y z}^k (x, y, z) \, dz \, dy +
\]

\[
+ \int \int_{\Omega^k}^{z_{\text{bot}_k}} [\delta u^k_z (x, y, z = z_{\text{bot}_k}) P_{z z}^k (x, y, z) \, dz \, dy +
\]

\[
+ \int \int_{\Gamma^k_{\text{top}}}^{z_{\text{top}_k}} [\delta u^k_n \sigma_{n n}^k + \delta u^k_s \sigma_{n s}^k + \delta u^k_s \sigma_{n z}^k] \, dz \, ds
\]

where \( P_{x z}^k, P_{y z}^k \) and \( P_{z z}^k \) are the top pressures (applied at \( z = z_{\text{top}_k} \)), \( P_{x z}^b, P_{y z}^b \) and \( P_{z z}^b \) are the bottom pressures (applied at \( z = z_{\text{bot}_k} \)), \( \sigma_{n n}^k, \sigma_{n s}^k \) and \( \sigma_{n z}^k \) are the specified normal and tangential components measured per unit area, \( u^k_n, u^k_s \) and \( u^k_z \) are the normal and tangential displacements on the edge \( \Gamma^k \) in which the stresses are specified. It is assumed that \( \Omega^k = \Omega \).

All the stresses and strains are retained. No restricting hypotheses are formulated. The relation between the stresses and strains is Hooke’s law (see Appendix A). The geometric relations relate the strains to the derivative of the displacements. Using the formalism of the Generalized Unified Formulation it is possible to write:

\[
\varepsilon_{x x}^k = \frac{\partial u^k_x}{\partial x} = x F_{\alpha u}^k u_{2 \alpha u_x}^k,
\]

\[
\varepsilon_{y y}^k = \frac{\partial u^k_y}{\partial y} = y F_{\alpha u}^k u_{y a y_y}^k
\]

\[
\gamma_{x y}^k = \frac{\partial u^k_x}{\partial y} + \frac{\partial u^k_y}{\partial x} = x F_{\alpha u}^k u_{2 \alpha u_y}^k + y F_{\alpha u}^k u_{y a y_x}^k
\]

\[
\gamma_{x z}^k = \frac{\partial u^k_x}{\partial z} + \frac{\partial u^k_z}{\partial x} = x F_{\alpha u}^k u_{2 \alpha u_z}^k + z F_{\alpha u}^k u_{2 \alpha u_x}^k
\]

\[
\gamma_{y z}^k = \frac{\partial u^k_y}{\partial z} + \frac{\partial u^k_z}{\partial y} = y F_{\alpha u}^k u_{y a z_y}^k + z F_{\alpha u}^k u_{2 \alpha u_y}^k
\]

\[
\varepsilon_{z z}^k = \frac{\partial u^k_z}{\partial z} = z F_{\alpha u}^k u_{2 \alpha u_z}^k
\]

Where the symbol “,” indicates the derivative. For example \( \gamma_{x y}^k \) indicates the derivative with respect to \( z \). The explicit form for the function \( \delta \varepsilon_k \frac{T}{p G} \sigma_{p H}^k \) under the sign of integral (see equation 10) is:
\[
\delta e_{pG}^{kT} \sigma_{pH}^{k} = \delta e_{zG}^{k} C_{11zG}^{k} + \delta e_{zzG}^{k} C_{12zzG}^{k} + \delta e_{xxG}^{k} C_{16}^{k} + \delta e_{zyG}^{k} C_{22zyG}^{k} + \delta e_{yyG}^{k} C_{22yyG}^{k} + \delta e_{zG}^{k} C_{26}^{k} + \delta e_{xG}^{k} C_{26}^{k} + \delta e_{G}^{k} C_{33}^{k} + \delta e_{zG}^{k} C_{36}^{k} + \delta e_{yG}^{k} C_{36}^{k} (12)
\]

Using the geometric relations (equation [11]):

\[
\delta e_{pG}^{kT} \sigma_{pH}^{k} = \tilde{C}_{11}^{k} F_{\alpha \alpha}^{z} F_{\beta \beta}^{z} \delta u_{\alpha \alpha_{zz}}^{k} u_{\beta \beta_{zz}}^{k} + \tilde{C}_{12}^{k} F_{\alpha \alpha}^{z} F_{\beta \beta}^{z} \delta u_{\alpha \alpha_{zz}}^{k} u_{\beta \beta_{zz}}^{k} + \tilde{C}_{16}^{k} F_{\alpha \alpha}^{z} F_{\beta \beta}^{z} \delta u_{\alpha \alpha_{zz}}^{k} u_{\beta \beta_{zz}}^{k} + \tilde{C}_{22}^{k} F_{\alpha \alpha}^{z} F_{\beta \beta}^{z} \delta u_{\alpha \alpha_{zz}}^{k} u_{\beta \beta_{zz}}^{k}
\]

\[
+ \tilde{C}_{26}^{k} F_{\alpha \alpha}^{z} F_{\beta \beta}^{z} \delta u_{\alpha \alpha_{zz}}^{k} u_{\beta \beta_{zz}}^{k} + \tilde{C}_{26}^{k} F_{\alpha \alpha}^{z} F_{\beta \beta}^{z} \delta u_{\alpha \alpha_{zz}}^{k} u_{\beta \beta_{zz}}^{k} + \tilde{C}_{26}^{k} F_{\alpha \alpha}^{z} F_{\beta \beta}^{z} \delta u_{\alpha \alpha_{zz}}^{k} u_{\beta \beta_{zz}}^{k} + \tilde{C}_{26}^{k} F_{\alpha \alpha}^{z} F_{\beta \beta}^{z} \delta u_{\alpha \alpha_{zz}}^{k} u_{\beta \beta_{zz}}^{k} + \tilde{C}_{26}^{k} F_{\alpha \alpha}^{z} F_{\beta \beta}^{z} \delta u_{\alpha \alpha_{zz}}^{k} u_{\beta \beta_{zz}}^{k} + \tilde{C}_{26}^{k} F_{\alpha \alpha}^{z} F_{\beta \beta}^{z} \delta u_{\alpha \alpha_{zz}}^{k} u_{\beta \beta_{zz}}^{k} (13)
\]

The function \( \delta e_{nG}^{k} \sigma_{nH}^{k} \) can be obtained by using a similar procedure:

\[
\delta e_{nG}^{kT} \sigma_{nH}^{k} = \delta e_{zzG}^{k} C_{11zG}^{k} + \delta e_{zzG}^{k} C_{12zzG}^{k} + \delta e_{xxG}^{k} C_{16}^{k} + \delta e_{yyG}^{k} C_{22yyG}^{k} + \delta e_{zG}^{k} C_{26}^{k} + \delta e_{xG}^{k} C_{26}^{k} + \delta e_{G}^{k} C_{33}^{k} + \delta e_{zG}^{k} C_{36}^{k} + \delta e_{yG}^{k} C_{36}^{k} (14)
\]

Using the geometric relations (equation [11]):

\[
\delta e_{nG}^{kT} \sigma_{nH}^{k} = \tilde{C}_{11}^{k} F_{\alpha \alpha}^{z} F_{\beta \beta}^{z} \delta u_{\alpha \alpha_{zz}}^{k} u_{\beta \beta_{zz}}^{k} + \tilde{C}_{12}^{k} F_{\alpha \alpha}^{z} F_{\beta \beta}^{z} \delta u_{\alpha \alpha_{zz}}^{k} u_{\beta \beta_{zz}}^{k} + \tilde{C}_{16}^{k} F_{\alpha \alpha}^{z} F_{\beta \beta}^{z} \delta u_{\alpha \alpha_{zz}}^{k} u_{\beta \beta_{zz}}^{k} + \tilde{C}_{22}^{k} F_{\alpha \alpha}^{z} F_{\beta \beta}^{z} \delta u_{\alpha \alpha_{zz}}^{k} u_{\beta \beta_{zz}}^{k}
\]

\[
+ \tilde{C}_{26}^{k} F_{\alpha \alpha}^{z} F_{\beta \beta}^{z} \delta u_{\alpha \alpha_{zz}}^{k} u_{\beta \beta_{zz}}^{k} + \tilde{C}_{26}^{k} F_{\alpha \alpha}^{z} F_{\beta \beta}^{z} \delta u_{\alpha \alpha_{zz}}^{k} u_{\beta \beta_{zz}}^{k} + \tilde{C}_{26}^{k} F_{\alpha \alpha}^{z} F_{\beta \beta}^{z} \delta u_{\alpha \alpha_{zz}}^{k} u_{\beta \beta_{zz}}^{k} + \tilde{C}_{26}^{k} F_{\alpha \alpha}^{z} F_{\beta \beta}^{z} \delta u_{\alpha \alpha_{zz}}^{k} u_{\beta \beta_{zz}}^{k} + \tilde{C}_{26}^{k} F_{\alpha \alpha}^{z} F_{\beta \beta}^{z} \delta u_{\alpha \alpha_{zz}}^{k} u_{\beta \beta_{zz}}^{k} + \tilde{C}_{26}^{k} F_{\alpha \alpha}^{z} F_{\beta \beta}^{z} \delta u_{\alpha \alpha_{zz}}^{k} u_{\beta \beta_{zz}}^{k} (15)
\]

Equations [13] and [15] (see also equation [10]) are integrated over the volume of layer \( k \). The integrals along the thickness can be immediately performed. The notation used for that integrals is shown in Figure.
Example 1: definition of 
\[ \zeta_{33}^{ka_{uz}} \beta_{uz} \]

\[ \zeta_{33}^{ka_{uz}} \beta_{uz} \] \[ \int_{z_{top}}^{z_{bot}} \zeta_{33}^{ka_{uz}} \beta_{uz} \, dz \]

Example 2: definition of 
\[ \zeta_{13}^{ka_{ux}} \beta_{uz} \]

\[ \zeta_{13}^{ka_{ux}} \beta_{uz} \] \[ \int_{z_{bot}}^{z_{top}} \tilde{\zeta}_{13}^{ka_{ux}} \beta_{uz} \, dz \]

Figure 3. GUF. Examples of some definitions related to the integrals along the thickness.

An integration by parts is performed (details omitted for brevity). Now the focus is on the external virtual work. Consider the applied pressures. They are written in a slightly different manner in order to use the power of the present Generalized Unified Formulation. Consider in particular the pressure term \( \delta u_k^x (x, y, z_{top}) P_{x}^{k \cdot t} (x, y, z_{top}) \) (see equation (16)). \( P_{x}^{k \cdot t} (x, y, z_{top}) \) is the force per unit of area in direction \( x \) (subscript \( x \)) and applied at the top surface \( (z = z_{top}) \) of the layer \( k \) (the superscript \( t \) means “top”). The pressure \( P_{x}^{k \cdot t} (x, y, z_{top}) \) is thought as a function along the thickness (this is in order to use GUF) as follows:

\[ P_{x}^{k \cdot t} (x, y, z_{top}) = \tilde{x} F_{\alpha}^{t} u_{\alpha} \cdot \tilde{x} F_{\beta}^{t} u_{\beta} P_{x}^{k \cdot t} \]  \( (16) \)

Notice that the same number of terms used for the expansion in the thickness direction of the displacement \( u_k^z \) has been considered. This is a natural choice, considering the fact that the pressure in \( x \) direction works with the displacement in \( x \) direction.

Now the following notation is introduced:

\[ \tilde{x} F_{\alpha}^{t} u_{\alpha} \cdot \tilde{x} F_{\beta}^{t} u_{\beta} \]  \( (17) \)

The superscript \( t \) means that the functions \( \tilde{x} F_{\alpha}^{t} u_{\alpha} \) are all calculated at the \( z \) which corresponds to the top surface of layer \( k \) (i.e., \( z = z_{top} \)) where the pressure is supposed to be applied. Using this definition, the pressure can be written by adopting the formalism already used for the displacements:

\[ P_{x}^{k \cdot t} (x, y, z_{top}) = \tilde{x} F_{\alpha}^{t} u_{\alpha} \cdot P_{x}^{k \cdot t} \]  \( (18) \)

Observing that

\[ \delta u_k^x (x, y, z_{top}) = \tilde{x} F_{\alpha}^{t} u_{\alpha} \cdot \delta u_{x}^{k \cdot t} \]  \( (19) \)

the contribution of the pressure \( P_{x}^{k \cdot t} (x, y, z_{top}) \) to the external virtual work can be written as (the secondary master index \( \beta \) needs to be used):

\[ \delta u_k^x (x, y, z_{top}) P_{x}^{k \cdot t} (x, y, z_{top}) = \delta u_k^x \tilde{x} F_{\alpha}^{t} u_{\alpha} \tilde{x} F_{\beta}^{t} u_{\beta} P_{x}^{k \cdot t} \]  \( (20) \)

or

\[ \delta u_k^x (x, y, z_{top}) P_{x}^{k \cdot t} (x, y, z_{top}) = \delta u_k^x \tilde{x} D_{\alpha}^{x} u_{\alpha} \tilde{x} F_{\beta}^{t} u_{\beta} P_{x}^{k \cdot t} \]  \( (21) \)

where

\[ \tilde{x} D_{\alpha}^{x} u_{\alpha} = \tilde{x} F_{\alpha}^{t} u_{\alpha} \]  \( (22) \)
The other applied pressures are similarly treated. Superscript \( b \) is used to indicate the bottom pressure; for example, \( \delta F_{\alpha \sigma}^b \nslant \delta F_{\alpha \sigma} \) (\( z = z_{bottom} \)). The virtual external work can then be rewritten as
\[
\delta L_{c}^k = + \int_{\Omega^k} \delta u_{x \alpha}^k \frac{t}{x \beta} D_{u_x u_x}^{k \alpha \beta} \frac{t}{x \beta} \, dx \, dy + \int_{\Omega^k} \delta u_{u_y}^k \frac{t}{y \beta} D_{u_y u_y}^{k \alpha \beta} \frac{t}{y \beta} \, dx \, dy \\
+ \int_{\Omega^k} \delta u_{x z}^k \frac{t}{x \beta} D_{u_x u_y}^{k \alpha \beta} \frac{b}{y \beta} \, dx \, dy + \int_{\Omega^k} \delta u_{y z}^k \frac{b}{x \beta} D_{u_y u_y}^{k \alpha \beta} \frac{b}{y \beta} \, dx \, dy \\
+ \int_{\Omega^k} \delta u_{y z}^k \frac{b}{x \beta} D_{u_y u_x}^{k \alpha \beta} \frac{b}{x \beta} \, dx \, dy + \int_{\Omega^k} \delta u_{x z}^k \frac{b}{x \beta} D_{u_y u_x}^{k \alpha \beta} \frac{b}{x \beta} \, dx \, dy \\
+ \int_{\Gamma_s^k} \left[ \delta u_{x \alpha}^k \sigma_{nn}^k + \delta u_{u_y}^k \sigma_{ns}^k + \delta u_{x z}^k \sigma_{nz}^k \right] \, dz \, ds \\
\tag{23}
\]

The pressure terms (for example \( P_{u_x u_x}^k \)) are inputs of the problem. Terms of the type \( \frac{t}{x \beta} D_{u_x u_x}^{k \alpha \beta} \) are defined as pressure fundamental kernels and are \( 1 \times 1 \) matrices. The pressure has been treated in a particular form because it is a natural choice for the case of multilayered structures analyzed with GUF.

The term in equation (23) which contains the prescribed stresses, is elaborated to obtain an expression that contains the displacements \( u_{x \alpha}^k, u_{y}^k \) and \( u_{z}^k \). This elaboration is required since, in general, different orders of expansions are used for different variables. After a few elaborations it can be demonstrated that the contribution of layer \( k \) to the expression of virtual external work is:
\[
\delta L_{c}^k = + \int_{\Omega^k} \delta u_{x \alpha}^k \frac{t}{x \beta} D_{u_x u_x}^{k \alpha \beta} \frac{t}{x \beta} \, dx \, dy + \int_{\Omega^k} \delta u_{u_y}^k \frac{t}{y \beta} D_{u_y u_y}^{k \alpha \beta} \frac{t}{y \beta} \, dx \, dy \\
+ \int_{\Omega^k} \delta u_{x z}^k \frac{t}{x \beta} D_{u_x u_y}^{k \alpha \beta} \frac{b}{y \beta} \, dx \, dy + \int_{\Omega^k} \delta u_{y z}^k \frac{b}{x \beta} D_{u_y u_y}^{k \alpha \beta} \frac{b}{y \beta} \, dx \, dy \\
+ \int_{\Omega^k} \delta u_{y z}^k \frac{b}{x \beta} D_{u_y u_x}^{k \alpha \beta} \frac{b}{x \beta} \, dx \, dy + \int_{\Omega^k} \delta u_{x z}^k \frac{b}{x \beta} D_{u_y u_x}^{k \alpha \beta} \frac{b}{x \beta} \, dx \, dy \\
+ \int_{\Gamma_s^k} \delta u_{x \alpha}^k \frac{k}{x \beta} \, dx \, ds + \int_{\Gamma_s^k} Z_{u_x u_y}^k \frac{k}{y \beta} \, dy \, ds \\
\tag{24}
\]

It is now assumed that where \( \frac{k}{x \beta} \) is assigned the corresponding quantity \( u_{x \alpha}^k \) is not assigned. Similarly, where \( \frac{k}{y \beta} \) is assigned \( u_{y}^k \) is not assigned and where \( \frac{k}{z \beta} \) is assigned \( u_{z}^k \) is not assigned. When the displacements are assigned, the virtual variations of the displacements are zero. For example, consider the displacement \( u_{x \alpha}^k \), which is assigned (see above) only on the boundary portion \( \Gamma_{x \alpha}^k = \Gamma_{x \alpha}^k \). Therefore, in this portion of the boundary \( \delta u_{x \alpha}^k = 0 \). Similar considerations can be made for the other displacements and, thus, the following relations can be written:
\[
\delta u_{x \alpha}^k = 0 \quad \text{on} \quad \Gamma_{x \alpha}^k - \Gamma_{x \alpha}^k \\
\delta u_{y}^k = 0 \quad \text{on} \quad \Gamma_{x \alpha}^k - \Gamma_{y \beta}^k \\
\delta u_{z}^k = 0 \quad \text{on} \quad \Gamma_{x \alpha}^k - \Gamma_{z \beta}^k \\
\tag{25}
\]

Considering these last relations and with the help of Figure 3, the governing equations are:
of Appendix A it is deduced that in the Navier-type solution only lamination schemes with angles 0 or 90 are used. Thus, from equation (26)

\[ \delta u_{x_{\alpha}}^k : = -Z_{11}^{\alpha x} u_{x_{\alpha}}^{k} - Z_{12}^{\alpha y} u_{y_{\beta}}^{k} - Z_{13}^{\alpha z} u_{z_{\gamma}}^{k} - Z_{16}^{\alpha x} u_{x_{x_{\alpha}}^{k}} - Z_{12}^{\alpha y} u_{y_{y_{\beta}}^{k}} - Z_{13}^{\alpha z} u_{z_{z_{\gamma}}^{k}} - Z_{16}^{\alpha x} u_{x_{x_{x_{\alpha}}^{k}}} = 0 \]  

(26)

\[ \delta u_{y_{\alpha}}^k : = -Z_{12}^{\alpha y} u_{y_{\alpha}}^{k} - Z_{22}^{\alpha y} u_{y_{y_{\beta}}^{k}} - Z_{23}^{\alpha y} u_{y_{z_{\gamma}}^{k}} - Z_{26}^{\alpha y} u_{y_{y_{y_{\beta}}^{k}}} - Z_{23}^{\alpha y} u_{y_{y_{z_{\gamma}}^{k}}} - Z_{26}^{\alpha y} u_{y_{y_{y_{y_{\beta}}^{k}}}} = 0 \]  

(27)

\[ \delta u_{z_{\alpha}}^k : = -Z_{25}^{\alpha z} u_{z_{\alpha}}^{k} - Z_{26}^{\alpha z} u_{y_{y_{\beta}}^{k}} - Z_{25}^{\alpha z} u_{y_{y_{y_{\beta}}^{k}}} - Z_{25}^{\alpha z} u_{z_{y_{z_{\gamma}}^{k}}} - Z_{26}^{\alpha z} u_{y_{y_{y_{y_{\beta}}^{k}}}} - Z_{25}^{\alpha z} u_{z_{y_{y_{z_{\gamma}}^{k}}}} = 0 \]  

(28)

Boundary conditions omitted for brevity. The governing equations here obtained are formally independent on the actual orders used to expand the displacements. This feature will lead to the invariant kernels from which all possible theories are generated.

**B. Navier-Type Solution**

To demonstrate how the invariant model is built, a Navier-type solution is considered. The invariant models are built under some hypothesis which can be easily removed when the Finite Element Method is used. The FEM procedure is not much different from a formal point of view and will be presented in future works.

In the Navier-type solution only lamination schemes with angles 0 or 90 are used. Thus, from equation 63 of Appendix A it is deduced that \( C_{16} = C_{26} = C_{16} = C_{45} = 0 \). Suppose also that the reference plane of the plate is a rectangle with length \( a \) in the \( x \) direction and \( b \) in the \( y \) direction. The external loads and displacements are assumed to have a sinusoidal distribution:
expansion (see equation 26). Similarly, consider equations 29 and 30 into relations 27 and 28 the governing equations become:

\[
\begin{align*}
\delta u_{xu_x} & : + K_{u_x u_x} \beta_x x U_{xu_x} + K_{u_x u_y} \beta_y y U_{yu_x} + K_{u_x u_z} \beta_z z U_{zu_x} = z R_{u_x u_x} \\
\delta u_{y \alpha y} & : + K_{u_y u_x} \beta_x x U_{xu_y} + K_{u_y u_y} \beta_y y U_{yu_y} + K_{u_y u_z} \beta_z z U_{zu_y} = y R_{u_x u_y} \\
\delta u_{z \alpha z} & : + K_{u_z u_x} \beta_x x U_{xu_z} + K_{u_z u_y} \beta_y y U_{yu_z} + K_{u_z u_z} \beta_z z U_{zu_z} = z R_{u_x u_z}
\end{align*}
\]

where the loads have been defined as follows:

\[
\begin{align*}
R_{u_x u_x} & = D_{u_x u_x} \beta_x x + P_{u_x u_x} \beta_x x + P_{u_x u_y} \beta_y y + P_{u_x u_z} \beta_z z \\
R_{u_x u_y} & = D_{u_x u_y} \beta_y y + P_{u_x u_x} \beta_x x + P_{u_x u_y} \beta_y y + P_{u_x u_z} \beta_z z \\
R_{u_x u_z} & = D_{u_x u_z} \beta_z z + P_{u_x u_x} \beta_x x + P_{u_x u_y} \beta_y y + P_{u_x u_z} \beta_z z
\end{align*}
\]

The 9 fundamental nuclei or kernels of the Generalized Unified Formulation (see equation 32) are defined as follows:

\[
\begin{align*}
K_{u_x u_x} \beta_x & = + 2 \frac{m \pi^2}{a^2} + Z_{66} u_x u_x \\
K_{u_y u_y} & = + \frac{m \pi^2}{ab} + Z_{66} u_y u_y \\
K_{u_z u_z} & = - \frac{m \pi}{a} + Z_{55} u_z u_z \\
K_{u_x u_y} & = Z_{12} u_x u_y \\
K_{u_x u_z} & = - Z_{12} u_x u_z \\
K_{u_y u_z} & = + Z_{12} u_y u_z \\
K_{u_y u_x} & = + \frac{m \pi^2}{ab} + Z_{66} u_y u_x \\
K_{u_z u_x} & = - \frac{m \pi}{a} + Z_{55} u_z u_x \\
K_{u_z u_y} & = Z_{22} u_z u_y \\
K_{u_y u_z} & = Z_{23} u_y u_z \\
K_{u_z u_x} & = + \frac{m \pi^2}{ab} + Z_{66} u_z u_x \\
K_{u_z u_y} & = - \frac{m \pi}{a} + Z_{55} u_z u_y \\
K_{u_z u_z} & = Z_{33} u_z u_z
\end{align*}
\]
But only six kernels are required to generate any theory. This can be seen from the following derivations. At structural level (after the assembling process) the governing equations can be written as:

\[
\begin{align*}
K_{u_x u_x} x U + K_{u_y u_y} y U + K_{u_z u_z} z U &= D_{u_x u_x} x P^t + D_{u_y u_y} y P^t + D_{u_z u_z} z P^t = z P \\
K_{u_x u_y} x U + K_{u_y u_y} y U + K_{u_x u_z} z U &= D_{u_x u_y} x P^t + D_{u_y u_y} y P^t + D_{u_x u_z} z P^t = y P \\
K_{u_x u_z} x U + K_{u_y u_y} y U + K_{u_z u_z} z U &= D_{u_x u_z} x P^t + D_{u_y u_z} y P^t + D_{u_z u_z} z P^t = x P
\end{align*}
\]  

(35)

In a matrix form, the system becomes:

\[
\begin{bmatrix}
K_{u_x u_x} & K_{u_y u_y} & K_{u_z u_z} \\
K_{u_x u_y} & K_{u_y u_y} & K_{u_x u_z} \\
\text{Symm} & K_{u_z u_z}
\end{bmatrix}
\begin{bmatrix}
\dot{x} U \\
\dot{y} U \\
\dot{z} U
\end{bmatrix} =
\begin{bmatrix}
\dot{z} P \\
\dot{y} P \\
\dot{x} P
\end{bmatrix}
\]

(36)

It is deduced (see equation [36]) that with the following six invariant fundamental kernels \( K_{u_x u_x} x z \), \( K_{u_y u_y} y z \), \( K_{u_z u_z} z z \), \( K_{u_x u_y} x y \), \( K_{u_y u_z} y z \), \( K_{u_x u_z} x z \) can generate all types of theories with any order of expansion in the thickness direction. Each variable can be represented with a different order with respect to the other variables. This property is also valid in the FEM representation.

The main properties of the Generalized Unified Formulation are described in Figure 4.

IV. \( \infty^3 \) Advanced Layerwise Theories Generated by Using the Generalized Unified Formulation

Equivalent Single Layer theories give a sufficiently accurate description of the global laminate response. However, these theories are not adequate for determining the stress fields at ply level. Layerwise theories assume separate displacement field expansions within each layer. The accuracy is then greater but the price is in the increased computational cost. Many layerwise plate models have been proposed in the past by applying Classical Plate Theory or Higher-order Shear Theories at each layer. Generalizations of these approaches were also given, and the displacements variables were expressed in terms of Lagrange polynomials. Many papers are devoted on the subject of layerwise theories. The conceptual differences between the displacement fields in layerwise and Equivalent Single Layer Theories are depicted in Figure 5. Layerwise models are computationally more expensive than the less accurate Equivalent Single Layer models. Therefore, layerwise models can be used in regions of the structure in which an accurate description is required, whereas Equivalent Single Layer models are employed in other parts of the structure. The Generalized Unified Formulation form for the displacements has been derived at layer level in equation 9. LWT are going to be developed and so equation 9 applies for this case. The functions of the thickness coordinate (see equation 9) are introduced in a general form. For example, \( F_i \) is a function of \( z \) and can be a polynomial, trigonometric, exponential or another function chosen a priori. To have the assembling process along the thickness direction immediate and intuitive and indicated for the case of multilayered structures a convenient expansion along the thickness is introduced.

The displacements must be continuous functions along the thickness to ensure the compatibility between two adjacent layers. Therefore, it is convenient for the axiomatic expansions along the thickness to have the following properties. The first property is that for \( z = z_{\text{bot}} \) (the bottom surface of layer \( k \)) all the functions along the thickness are zero except the one which multiplies the term corresponding to the bottom (subscript \( b \)). For example, in the case of the displacement \( u^k \), the functions calculated at the bottom of layer \( k \) should give the following values:

\[
\begin{align*}
\dot{x} F_i (z = z_{\text{bot}_k}) &= 0 \\
\dot{y} F_i (z = z_{\text{bot}_k}) &= 0 \\
\dot{z} F_b (z = z_{\text{bot}_k}) &= 1
\end{align*}
\]  

(37)

If the previous conditions are satisfied then \( u^k_{z_{\text{bot}_k}} \) is not just a term in the thickness expansion of the variable \( u^k \) but assumes the meaning of the value that the displacement \( u^k \) takes when the bottom surface of layer \( k \) is considered (i.e., \( z = z_{\text{bot}_k} \)). This now explains why the subscript “\( b \)” is introduced in the notation. Similar concept is applied for the top surface of the layer. A good set of function satisfies the following relations:

\[
\begin{align*}
\dot{x} F_i (z = z_{\text{top}_k}) &= 1 \\
\dot{y} F_i (z = z_{\text{top}_k}) &= 0 \\
\dot{z} F_b (z = z_{\text{top}_k}) &= 0
\end{align*}
\]  

(38)
Figure 4. Generalized Unified Formulation for displacement-based theories.

Lyerwise theories

Equivalent single layer theories

Figure 5. Layerwise theories vs equivalent single layer theories in a three layered structure.
and so $u^{k}_{x}$ is not just a term in the thickness expansion of the variable $u^{k}_{z}$ but assumes the meaning of the value that the displacement $u^{k}_{z}$ takes when the top surface of layer $k$ is considered (i.e., $z = z_{\text{top}_k}$). This now explains why the subscript “$t$” is introduced. The second property is that numerical stability (no ill conditioning when the orders are increased) is important. Therefore, orthogonal polynomials should be preferred.

A good set of functions which satisfy the above mentioned properties should be selected. It is possible to demonstrate that all the previous properties are satisfied if particular combination of Legendre polynomials is used. Legendre polynomials are defined in the interval $[-1,1]$. Thus, a transformation is necessary:

$$\zeta_{k} = \frac{2}{z_{\text{top}_k} - z_{\text{bot}_k}}z - \frac{z_{\text{top}_k} + z_{\text{bot}_k}}{z_{\text{top}_k} - z_{\text{bot}_k}} z_{\text{top}_k} - z_{\text{bot}_k}$$

where $\zeta_{k}$ is a non-dimensional coordinate. The following formula is also valid:

$$z_{\text{top}_k} = z_{\text{bot}_k} + h_{k}$$

The transformation is then

$$\zeta_{k} = \frac{2}{h_{k}} z - \frac{z_{\text{top}_k} + z_{\text{bot}_k}}{z_{\text{top}_k} - z_{\text{bot}_k}}$$

The Legendre polynomial of order zero is $P_{0}(\zeta_{k}) = 1$. The Legendre polynomial of order one is $P_{1}(\zeta_{k}) = \zeta_{k}$. The higher order polynomials can be obtained by using Bonnet’s recursion:

$$P_{n+1}(\zeta_{k}) = \frac{(2n+1) \zeta_{k} P_{n}(\zeta_{k}) - n P_{n-1}(\zeta_{k})}{n+1}$$

Bonnet’s formula is a convenient method to calculate the Legendre polynomials in a practical code based on the Generalized Unified Formulation.

The same type of functions for all displacements are used. This is not necessary with the Generalized Unified Formulation but it is more practical. The following combination of Legendre functions is used:

$$x F_{l} = y F_{l} = z F_{l} = \frac{P_{l} + P_{l-2}}{2}, \quad x F_{b} = y F_{b} = z F_{b} = \frac{P_{b} + P_{b-2}}{2}$$

$$x F_{m} = P_{m - P_{m-2}}, \quad m = 2, 3, ..., N_{u_{z}}$$

$$y F_{m} = P_{m - P_{m-2}}, \quad m = 2, 3, ..., N_{u_{y}}$$

$$z F_{n} = P_{n - P_{n-2}}, \quad n = 2, 3, ..., N_{u_{z}}$$

in which $P_{j} = P_{j}(\zeta_{k})$ is the Legendre polynomial of $j$-order. The chosen functions have the following properties:

$$\zeta_{k} = \begin{cases} +1, & x F_{l}, y F_{l}, z F_{l} = 1, \quad x F_{b}, y F_{b}, z F_{b} = 0, \quad x F_{m}, y F_{m}, z F_{m} = 0 \\ -1, & x F_{l}, y F_{l}, z F_{l} = 0, \quad x F_{b}, y F_{b}, z F_{b} = 1, \quad x F_{m}, y F_{m}, z F_{m} = 0 \end{cases}$$

Thus, the properties earlier mentioned are all satisfied and this set of functions is a good choice to build the advanced layerwise theories. It is then possible to create any class of theories by changing the orders of displacements. Suppose, for example, that a theory has the following data: $N_{u_{z}} = 3, N_{u_{y}} = 2, N_{u_{z}} = 4$. The corresponding theory is indicated as $LD_{324}$. The first letter “L” means “Layerwise” theory, the second letter “D” means that a “Displacement-based” theory is formulated (i.e. the variational statement is the Principle of Virtual Displacements) The subscripts are the orders of the Legendre polynomials used for the displacements. In general, the acronym is then built as follows: $LD_{N_{u_{y}}, N_{u_{y}}, N_{u_{z}}}$.

The logic used to build the acronyms and one example are shown in Figure 33.

A. Expansion of the Six $1 \times 1$ Invariant Kernels: Matrices at Layer Level

In this paper the expansion used in the different variables does not change and each layer is treated in the same way. Thus, for example, $N_{u_{z}} = N_{u_{z+1}} = N_{u_{z}}$. The expansion of the kernels is the most important part of the generation of one of the possible $\infty^3$ layerwise theories. This operation is done at layer level. To
Advanced Layerwise Theories (ALWT)

<table>
<thead>
<tr>
<th>Meaning of the acronym</th>
<th>LD ( N_{uz}N_{uy}N_{uz} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L ) = Layewrwise</td>
<td>( u^s ) = Order ( N_{ux} ) (expansion along the thickness)</td>
</tr>
<tr>
<td>( D ) = Displacement-based theory</td>
<td>( u^s ) = Order ( N_{ux} ) (expansion along the thickness)</td>
</tr>
<tr>
<td></td>
<td>( u^s ) = Order ( N_{ux} ) (expansion along the thickness)</td>
</tr>
</tbody>
</table>

**Example: theory \( LD_{132} \)**

\[
\begin{align*}
\Psi^1_x &= \frac{\partial^2 \phi}{\partial x^2} u^1_x + \frac{\partial \phi}{\partial z} u^1_z \\
\Psi^2_x &= \frac{\partial^2 \phi}{\partial x^2} u^2_x + (P_2 - P_1) u^2_z + \frac{\partial \phi}{\partial z} u^2_z \\
\Psi^3_x &= \frac{\partial^2 \phi}{\partial x^2} u^3_x + (P_2 - P_1) u^3_z + \frac{\partial \phi}{\partial z} u^3_z \\
\Psi^4_x &= \frac{\partial^2 \phi}{\partial x^2} u^4_x + (P_2 - P_1) u^4_z + \frac{\partial \phi}{\partial z} u^4_z
\end{align*}
\]

Or

\[
\begin{align*}
u^1_x &= T_{ux} u^1_x a_n = t, b \\
u^2_x &= T_{ux} u^2_x a_n = t, m, b \quad m = 2, 3 \\
u^3_x &= T_{ux} u^3_x a_n = t, n, b \quad n = 2
\end{align*}
\]

Figure 6. Acronyms used to define the Advanced Layerwise Theories using the Generalized Unified Formulation.

explain how this operation is performed, consider the case of theory \( LD_{326} \), in which the number of Degrees of Freedom, at layer level, is the following:

\[
\begin{align*}
[DOF]_{ux}^k &= N_{ux} + 1 = 3 + 1 = 4 \\
[DOF]_{uz}^k &= N_{uz} + 1 = 2 + 1 = 3 \\
[DOF]_{uz}^k &= N_{uz} + 1 = 4 + 1 = 7
\end{align*}
\] (45)

From the number of Degrees of Freedom it is possible to calculate the size of the layer matrices. For example, when matrix \( K_{ux}^{k \alpha \beta \gamma} u_{ux} \) is expanded then the final size at layer level will be \( [DOF]_{ux}^k \times [DOF]_{uz}^k \). In the example relative to theory \( LD_{326} \), matrix \( K_{ux}^{k \alpha \beta \gamma} u_{ux} \) at layer level (indicated as \( K_{ux}^{k \alpha \beta \gamma} u_{ux} \)) is a 4 \( \times \) 7 matrix and obtained as explained in Figure 7.
Advanced Layerwise Theory $LD_{326}$

Figure 7. Generalized Unified Formulation: example of expansion from a kernel to a layer matrix. Case of theory $LD_{326}$. From $K_{u_x u_z}^{k, \alpha \beta}$ to $K_{u_x u_z}^{k}$. 

$$
K_{u_x u_z}^{k} = N_{u_x} + 1 = 4
$$

$$
K_{u_x u_z}^{k, \alpha \beta} = N_{u_x} + 1 = 7
$$

$$
[DOF]_{u_x}^k = [DOF]_{u_x}^{k, \alpha \beta}
$$

Expanded matrix (layer level)
B. Assembling in the Thickness Direction: from Layer to Multilayer Level

The assembling must consider the compatibility of the displacements between two adjacent layers. Figure 8 shows how the assembling of a typical matrix is performed. The pressure matrices are obtained from the pressure kernels using the same method shown in Figures 7 and 8. The use of combinations of Legendre polynomials ensures the continuity of the functions, which are used to expand the displacements. Figure 9 shows this concept for the case of theory LD.324. The pressure amplitudes at multilayer level are input of the problem. Some input examples are shown in Figure 10. The layer pressure matrices (for example $D_k^{(k+1)} u_x u_z$) are obtained using a method similar to the one adopted for the stiffness matrices. They are generated from the pressure fundamental kernels and assembled. In the practice, all the pressure matrices are set to be with only zeros except the top and bottom layers in which the pressure is actually applied. Once the matrices are all assembled, equation 36 is obtained and the displacement amplitudes can be found. In the case of FEM applications the unknown will be the nodal displacements and the concepts are the same.

Figure 8. Generalized Unified Formulation: example of assembling from layer matrices to multilayer matrix. Case of theory LD.326. From $K_{u_x u_z}^{(k+1)}$ and $K_{u_x u_z}^{(k)}$ to $K_{u_x u_z}^{(k+1)}$.
Advanced Layerwise Theory \( \textit{LD}_{324} \)

This example assumes two layers

\[
[D\text{OF}]_{\text{u,}1} = N_u + 1 = 4 \Rightarrow [D\text{OF}]_{\text{u,}2} = [D\text{OF}]_{\text{u,}1} \cdot (N_l - (N_l - 1)) = 7
\]

\[
[D\text{OF}]_{\text{u,}2} = N_u + 1 = 3 \Rightarrow [D\text{OF}]_{\text{u,}3} = [D\text{OF}]_{\text{u,}2} \cdot (N_l - (N_l - 1)) = 5
\]

\[
[D\text{OF}]_{\text{u,}3} = N_u + 1 = 5 \Rightarrow [D\text{OF}]_{\text{u,}4} = [D\text{OF}]_{\text{u,}3} \cdot (N_l - (N_l - 1)) = 9
\]

Unknown displacement amplitudes

\[
\begin{aligned}
&x U^{(k+1)} = x U^k & y U^{(k+1)} = y U^k & z U^{(k+1)} = z U^k \\
&x U^k & y U^k & z U^k
\end{aligned}
\]

Figure 9. Case of theory \( LD_{324} \). Multilayer unknown displacement and out-of-plane stresses for the case in which the number of layers is two.

V. \( \infty^3 \) Advanced Higher-order Shear Deformation Theories Generated by Using the Generalized Unified Formulation

The derivation is started by considering a particular theory in which the in-plane displacements are expanded along the thickness by using a cubic polynomial and the out-of-plane displacement \( u_z \) is parabolic. In this case it is possible to write the displacements as follows:

Theory I:

\[
\begin{aligned}
&u_x = u_{x0} + z \phi_{1u_x} + z^2 \phi_{2u_x} + z^3 \phi_{3u_x} \\
&u_y = u_{y0} + z \phi_{1u_y} + z^2 \phi_{2u_y} + z^3 \phi_{3u_y} \\
&u_z = u_{z0} + z \phi_{1u_z} + z^2 \phi_{2u_z}
\end{aligned}
\]

(46)

For each displacement component the concepts of the Generalized Unified Formulation can be applied. For example, the displacement \( u_x \) is written as

\[
u_x = u_{x0} + z \phi_{1u_x} + z^2 \phi_{2u_x} + z^3 \phi_{3u_x} = ^x F_1 u_{x0} + ^x F_2 u_{x2} + ^x F_3 u_{x3} + ^x F_b u_{xb}
\]

(47)

where

\[
^x F_1 = 1; \quad ^x F_2 = z; \quad ^x F_3 = z^2; \quad ^x F_b = z^3
\]

(48)

\[
u_{x0} = u_{x0}; \quad u_{x2} = \phi_{1u_x}; \quad u_{x3} = \phi_{2u_x}; \quad u_{xb} = \phi_{3u_x}
\]

The GUF for the displacement \( u_x \) is:

\[
u_x = x F_{a_{ux}} u_{x_{a_{ux}}} \quad \alpha_{ux} = l, b; \quad l = 2, ..., N_{a_{ux}}
\]

(49)

where, in the example, \( N_{a_{ux}} = 3 \). Notice that the superscript "x" in \( x F_{a_{ux}} \) is used to clearly enhance that the displacement \( u_x \) (displacement in the \( x \) direction) is being considered.

For the displacement \( u_y \) the formal procedure produces similar results (notice that now the slave index is \( u_y \)):

\[
u_y = y F_{a_{uy}} u_{y_{a_{uy}}} \quad \alpha_{uy} = m, b; \quad m = 2, ..., N_{a_{uy}}
\]

(50)
Advanced Layerwise Theory $LD_{324}$

This example assumes two layers

\[
[DOF]_{w_t}^k = N_w + 1 = 4 \Rightarrow [DOF]_{w_t}^k = [DOF]_{w_t}^k \cdot N_t - (N_t - 1) = 7
\]

\[
[DOF]_{w_t}^k = N_w + 1 = 3 \Rightarrow [DOF]_{w_t}^k = [DOF]_{w_t}^k \cdot N_t - (N_t - 1) = 5
\]

\[
[DOF]_{w_t}^k = N_w + 1 = 5 \Rightarrow [DOF]_{w_t}^k = [DOF]_{w_t}^k \cdot N_t - (N_t - 1) = 9
\]

Figure 10. Case of theory $LD_{324}$. Example of pressure amplitudes and inputs at multilayer level for the case in which the number of layers is two.
The displacement $u_z$ is only parabolic (three terms are used in the expansion), but the representation is formally the same:

$$u_z = F_{n_u} \alpha_{u_z} \quad \alpha_{u_z} = t, n, b; \quad n = 2, ..., N_{u_z}$$ (51)

The superscript $k$ is not present, whereas in the layerwise case it was. In fact, in Equivalent Single Layer Models the displacement fields have a description at plate level and not at layerwise level. The Generalized Unified Formulation for this case (equations [49], [50], and [51]) is formally equivalent to the writing of equation [9]. This similarity suggests that it is possible to use the layerwise Generalized Unified Formulation for the Equivalent Single Layer case too. That is, equation [9] can be used for the Equivalent Single Layer case. The fact that the displacement field does not have a layerwise description (see, for example, Theory I explicitly written in equation [46]) is taken into account when the assembling in the thickness direction of the layer matrices is considered. For the Advanced Higher-order Shear Deformation Theories it is then possible to use equation [9]. However, now the functions of the thickness coordinates are going to be different. The functions of the thickness coordinate are assumed to be of the type $1, z, z^2, z^3, ...,$. This choice is made for “consistency” with the usual approach used in the literature for Equivalent Single Layer Models. The actual functions are reported below:

$$F_1 = 1, \quad F_1 = 1, \quad F_1 = 1,$$

$$F_2 = z, \quad F_2 = z, \quad F_2 = z,$$

$$F_3 = z^2, \quad F_3 = z^2, \quad F_3 = z^2,$$

$$...$$

$$F_m = z^{m-1}, \quad F_m = z^{m-1}, \quad F_m = z^{m-1},$$

$$...$$

$$F_n = z^{N_{u_z}}, \quad F_n = z^{N_{u_y}}, \quad F_n = z^{N_{u_z}}.$$ (52)

The Advanced Higher-order Shear Deformation Theories (AHSDT) are indicated with acronyms shown in Figure 11. As for the layerwise case it is possible to create a class of theories by changing the order used for the displacements. Suppose, for example, that a theory has the following data: $N_{u_x} = 3, N_{u_y} = 2, N_{u_z} = 4$. The corresponding AHSDT theory is indicated as $ED_{132}$. The first letter “E” means Equivalent Single Layer theory, the second letter “D” means that a displacement-based formulation is used. The subscripts are the orders of the polynomials used for the displacements. In general, the acronym is then built as follows: $ED_{N_{u_x}, N_{u_y}, N_{u_z}}$.
AHSĐT $ED_{324}$

$$[DOF]_{u_z} = N_{u_z} + 1 = 4 \quad [DOF]_{u_y} = N_{u_y} + 1 = 3$$

If we have $N_l$ layers, the number of Degrees of Freedom is obtained as follows:

$$[DOF]_{u_z} = [DOF]_{u_y}^k \text{ (ESL description!)}$$

Layers have different thicknesses and material properties. So the matrices are different

$$K_{u_z u_y}^{(k+1)} \neq K_{u_z u_y}^k$$

Two layers are assumed in this example. So we have $[DOF]_{u_z} = 4$ \quad $[DOF]_{u_y} = 3$

---

**A. Expansion of the Six $1 \times 1$ Invariant Kernels: Matrices at Layer Level**

Besides the different form of the functions used to expand the displacements, AHSĐT do not present formal differences with respect to the laye-wise cases. For example, the generation of layer matrix $K^k_{u_z u_z}$ from the kernel $K_{u_z u_z}^{h_1 h_2}$ in the cases of theories $LD_{326}$ and $ED_{326}$ (which have the same number of Degrees of Freedom at layer level) is formally the same. Thus, Figure 7 applies for the case of theory $ED_{326}$ as well. Of course, the actual integrals along the thickness are different because different functions have been used (compare equations 52 and 43 and see the definitions of Figure 8).

**B. Assembling in the Thickness Direction: from Layer to Multilayer Level**

The displacement fields are treated as equivalent single layers quantities, and this makes the assembling procedure different with respect to the layerwise case. The continuity of the displacements in the thickness direction must be imposed. So, it is not difficult to show that the assembling is performed as shown in Figure 12. The pressure matrices are obtained using similar procedure and the details are omitted for brevity. About the pressure amplitudes, inputs of the problem (see Figure 13), the difference between
this ESL case and the corresponding Layerwise case (Figure 10) should be noted. The final equations at

AHSDT \textbf{ED}_{324}

In this example two layers are assumed

\[
[\text{DOF}]_a = [\text{DOF}]_b = N_a + 1 = 4
\]

\[
[\text{DOF}]_a = [\text{DOF}]_b = N_a + 1 = 3
\]

\[
[\text{DOF}]_a = [\text{DOF}]_b = N_a + 1 = 5
\]

\begin{tabular}{|c|c|}
\hline
\hline
Top pressures & Bottom pressures \\
\hline
\hline
\[ x \mathbf{P}^{t} \] & \[ x \mathbf{P}^{b} \] \\
\hline
\[ 1 \] & \[ 0.5 \] \\
\[ 0 \] & \[ 0 \] \\
\[ 0 \] & \[ 0 \] \\
\hline
\[ y \mathbf{P}^{t} \] & \[ y \mathbf{P}^{b} \] \\
\hline
\[ 5 \] & \[ 2 \] \\
\[ 0 \] & \[ 0 \] \\
\[ 0 \] & \[ 0 \] \\
\hline
\[ z \mathbf{P}^{t} \] & \[ z \mathbf{P}^{b} \] \\
\hline
\[ 4 \] & \[ 3 \] \\
\[ 0 \] & \[ 0 \] \\
\[ 0 \] & \[ 0 \] \\
\hline
\end{tabular}

In this example the following case is considered (pressures applied at the top surface of layer 2):

\[ P_{x}^{t} = 3 \cos \frac{m_{\sigma}}{a} \sin \frac{ny}{b} \]

\[ P_{y}^{t} = 5 \sin \frac{m_{\sigma}}{a} \cos \frac{ny}{b} \]

\[ P_{z}^{t} = -4 \sin \frac{m_{\sigma}}{a} \sin \frac{ny}{b} \]

In this example the following case is considered (pressures applied at the bottom surface of layer 1):

\[ P_{x}^{b} = 0.5 \cos \frac{m_{\sigma}}{a} \sin \frac{ny}{b} \]

\[ P_{y}^{b} = 2 \sin \frac{m_{\sigma}}{a} \cos \frac{ny}{b} \]

\[ P_{z}^{b} = 3 \sin \frac{m_{\sigma}}{a} \sin \frac{ny}{b} \]

Figure 13. Case of theory \textit{ED}_{324}. Example of pressure amplitudes and inputs at multilayer level for the case in two-layered case.

multilayer level are again given by relation (30). This equivalence is another advantage of the Generalized Unified Formulation.

**VI. Advanced Higher-order Shear Deformation Theories with Zig-Zag Effects Generated by Using the Generalized Unified Formulation**

The Higher-order Shear Deformation Theories may be not sufficiently accurate in some challenging cases. One option would be to abandon the Equivalent Single Layer Description and use Layerwise Theories. However, this may be computationally too expensive. Is there anything more accurate than HSCTD with a similar number of Degrees of Freedoms? The answer is yes, zig-zag theories.\(^{30}\) The concept behind zig-zag theories and zig-zag form of the displacements is the following. The equilibrium between two adjacent layers implies that the out-of-plane stresses are equal at the interface. These stresses can be thought as a combination of strains multiplied by some coefficients that depend on the material of each layer (Hooke’s law). In general, two layers have different mechanical properties and, therefore, different strains are required to obtain equilibrium. The strains are related to the derivatives of the displacements (geometric relations). Thus, different strains imply different slopes of the displacements. This fact leads to the zig-zag form of the displacements (see Figure 14).
A very large amount of literature has been devoted to the formulation of axiomatic zig-zag theories that take into account these requirements. Three different categories of zig-zag theories can be created. The first category is Lekhnitskii Multilayered Theory (LMT). The second is Ambartsumian Multilayered Theory (AMT) and the third is Reissner Multilayered Theory (RMT). LMT was introduced for the particular case of cantilevered multilayered beam and almost ignored in subsequent works with a few exceptions. A summary of the main facts of LMT is presented in reference [30]. Ambartsumian work was an extension of Reissner-Mindlin theory. RMT is based on Reissner’s Mixed Variational Theorem (RMVT).

The present paper takes into account the Zig-Zag effects by adopting Murakami’s Zig-Zag Function (see Figure 14). MZZF has the advantage of being simple and reproducing the discontinuity of the first derivative of the displacements in the thickness direction. It will be demonstrated that the usage of MZZF is more effective than increasing the orders of the expansions of the variables along the thickness.

The derivation is started from the Advanced Higher-order Shear Deformation Theory expressed by equation 46. Theory I (see equation 46) is then “improved” as follows (Theory II is the resulting theory):

Theory II:

\[
\begin{align*}
\zeta_k &= \frac{2}{z_{\text{top}_k} - z_{\text{bot}_k}} \cdot z - \frac{z_{\text{top}_k} + z_{\text{bot}_k}}{z_{\text{top}_k} - z_{\text{bot}_k}} \\
\end{align*}
\]

The quantity \(\zeta_k\) is defined in equation 39. In equation 53, valid for a theory with Zig-Zag form of the displacements included, the following can be observed. In Murakami’s Zig-Zag functions the term \((-1)^k\) is present. \(k\) is the integer representing the ID of a generic layer, \(k=1\) is for the bottom layer and \(k=N_l\) is for the top layer (\(N_l\) is the number of layers). The term \((-1)^k\) enforces the discontinuity of the first derivative (thickness direction) of the displacement. For example, in layer \(k\) the derivative with respect to \(z\) of the Zig-Zag term relative to the component \(u_x\) is

\[
\frac{d}{dz} \left[ (-1)^k \zeta_k u_{xz} \right] = (-1)^k u_{xz} \frac{d\zeta_k}{dz} = (-1)^k u_{xz} \frac{2}{z_{\text{top}_k} - z_{\text{bot}_k}}
\]
As can be seen the term \((-1)^k\) strongly affects the sign of the derivative. The displacements still have an Equivalent Single Layer description. In fact, the terms \(u_{xz}\), \(u_{yz}\) and \(u_{zz}\) are independent on the actual layer and defined for the whole plate. The zig-zag form of the displacements is taken into account a priori by adding only three Degrees of Freedom \((u_{xz}, u_{yz} \text{ and } u_{zz})\). This is a general property and does not depend on the orders used for the expansion of the different variables. That is, a generic theory can take into account the zig-zag form of the displacements by adding only three extra Degrees of Freedom. Suppose, for example, that a theory has the Advanced Higher Order Shear Deformation Theories presented above. This is also evident if equations [49] and [50] for the case of Theory I. For each displacement component the concepts of the Generalized Unified Formulation is still valid. However, in this case equations [49], [50] and [51] have to be slightly modified because they have to contain an extra term which comes from the Zig-Zag Function. This is explained if for example displacement \(u_x\) is considered:

\[
\begin{align*}
  u_x &= u_{x0} + z\phi_{1x} + z^2\phi_{2x} + z^3\phi_{3x} + (-1)^k \zeta_k u_{xz} \\
  &= xF_1 u_{x1} + xF_2 u_{x2} + xF_3 u_{x3} + xF_4 u_{x4} + xF_5 u_{x5} \\
\end{align*}
\]

where

\[
xF_1 = 1; \quad xF_2 = z; \quad xF_3 = z^2; \quad xF_4 = z^3; \quad xF_5 = (-1)^k \zeta_k
\]

\[
\begin{align*}
  u_{x1} &= u_{x0}; \quad u_{x2} = \phi_{1x}; \quad u_{x3} = \phi_{2x}; \quad u_{x4} = \phi_{3x}; \quad u_{x5} = u_{xz}
\end{align*}
\]

The GUF for the displacement \(u_x\) is:

\[
\begin{align*}
  u_x &= xF_{a_{ux}} u_{x0} + xF_{b_{ux}} \quad \alpha_u = t, l, b; \quad l = 2, ..., N_{ux} + 1
\end{align*}
\]

where, in the example, \(N_{ux} = 3\). For the displacement \(u_y\) the formal procedure produces similar results:

\[
\begin{align*}
  u_y &= yF_{a_{uy}} u_{y0} + yF_{b_{uy}} \quad \alpha_u = t, m, b; \quad m = 2, ..., N_{uy} + 1
\end{align*}
\]

The displacement \(u_z\) is only parabolic, but the representation is formally the same:

\[
\begin{align*}
  u_z &= zF_{a_{uz}} u_{z0} + zF_{b_{uz}} \quad \alpha_u = t, n, b; \quad n = 2, ..., N_{uz} + 1
\end{align*}
\]

For the case of Zig-Zag theories, the Generalized Unified Formulation (equations [52], [53] and [54]) is formally equivalent to a layerwise case in which the same number of terms is considered. Thus (see equation [9]),

\[
\begin{align*}
  u_x^k &= xF_{a_{ux}} u_{x0}^k + xF_{b_{ux}} \quad \alpha_u = t, l, b; \quad l = 2, ..., N_{ux} + 1
\end{align*}
\]

\[
\begin{align*}
  u_y^k &= yF_{a_{uy}} u_{y0}^k + yF_{b_{uy}} \quad \alpha_u = t, m, b; \quad m = 2, ..., N_{uy} + 1
\end{align*}
\]

\[
\begin{align*}
  u_z^k &= zF_{a_{uz}} u_{z0}^k + zF_{b_{uz}} \quad \alpha_u = t, n, b; \quad n = 2, ..., N_{uz} + 1
\end{align*}
\]

where

\[
\begin{align*}
  xF_1 &= 1; \quad yF_1 = 1; \quad zF_1 = 1 \\
  xF_2 &= z; \quad yF_2 = z; \quad zF_2 = z \\
  xF_3 &= z^2; \quad yF_3 = z^2; \quad zF_3 = z^2 \\
  &\vdots \\
  xF_l &= z^{l-1}; \quad yF_m = z^{m-1}; \quad zF_n = z^{n-1} \\
  &\vdots \\
  xF_b &= (-1)^k \zeta_k; \quad yF_b = (-1)^k \zeta_k; \quad zF_b = (-1)^k \zeta_k
\end{align*}
\]

The Advanced Zig-Zag Theories (AZZT) are obtained by adding the zig-zag form of the displacements to the Advanced Higher Order Shear Deformation Theories presented above. This is also evident if equations [52] and [53] are compared. How the acronym are built is explained in Figure [15]. It is possible to create a class of theories by changing the order used for the displacements. Suppose, for example, that a theory has the following data: \(N_{ux} = 3\), \(N_{uy} = 2\), \(N_{uz} = 4\). The corresponding AZZT theory is indicated as \(EDZ_{324}\).
**Advanced Zig-Zag Theories**

**AZZT**

First letter “E” means “Equivalent Single Layer” theory, the second letter “D” means displacement-based theory, the third letter “Z” means that the zig-zag form of the displacements is enforced a priori. The subscripts are the orders of the polynomials used for the displacements. In general the acronym is then built as follows: $EDZ_{N_yN_zN_x}$. The formal generation of the stiffness matrices from the kernels of the Generalized Unified Formulation is done as in the case of AHSDT. The difference is only in the sizes of the matrices and of course in the actual values of the integrals along the thickness. For example, consider theories $ED{3_24}$ and $EDZ{3_24}$. They both have the same orders. However, in the Zig-Zag theory there is an extra degree of freedom for each displacement variable. This means that the layer matrix $K^k_{u_x,u_y}$ has size $4 \times 3$ in the case of theory $ED{3_24}$ whereas the size is $5 \times 4$ in the case of theory $EDZ{3_24}$. The assembling at multilayered level follows the same procedure outlined in the case of AHSDT. The final system that has to be solved is again equation [36].

**Appendix A. Coefficients of Hooke’s Law**

\[
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy} \\
\sigma_{xz} \\
\sigma_{yz} \\
\sigma_{zz}
\end{bmatrix} =
\begin{bmatrix}
\tilde{C}_{11} & \tilde{C}_{12} & \tilde{C}_{16} & 0 & 0 & \tilde{C}_{13} \\
\tilde{C}_{12} & \tilde{C}_{22} & \tilde{C}_{26} & 0 & 0 & \tilde{C}_{23} \\
\tilde{C}_{16} & \tilde{C}_{26} & \tilde{C}_{66} & 0 & 0 & \tilde{C}_{36} \\
0 & 0 & 0 & \tilde{C}_{55} & \tilde{C}_{45} & 0 \\
0 & 0 & 0 & \tilde{C}_{45} & \tilde{C}_{44} & 0 \\
\tilde{C}_{13} & \tilde{C}_{23} & \tilde{C}_{36} & 0 & 0 & \tilde{C}_{33}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\gamma_{xy} \\
\gamma_{xz} \\
\gamma_{yz} \\
\varepsilon_{zz}
\end{bmatrix}
\]  

(62)
where
\[
\begin{align*}
\tilde{C}_{11} &= C^{24}C_{11} + 2S^{2}C^{2}C_{12} + S^{4}C_{22} + 4S^{3}C^{2}C_{66} \\
\tilde{C}_{12} &= S^{2}C^{2}C_{11} + C^{4}C_{12} + S^{4}C_{12} + S^{2}C^{2}C_{22} - 4S^{3}C^{2}C_{66} \\
\tilde{C}_{16} &= SC^{2}C_{11} + S^{2}C_{12} - SC^{3}C_{12} - S^{3}C_{22} - 2SC^{5}C_{66} + 2S^{3}C^{2}C_{66} \\
\tilde{C}_{13} &= C^{2}C_{13} + C^{2}C_{2} \\
\tilde{C}_{22} &= S^{2}C^{2}C_{11} + 2S^{2}C^{2}C_{12} + C^{4}C_{22} + 4S^{2}C^{2}C_{66} \\
\tilde{C}_{26} &= S^{2}C^{4}C_{11} + SC^{3}C_{12} - SC^{3}C_{12} - 2SC^{5}C_{66} - 2SC^{3}C_{66} \\
\tilde{C}_{23} &= S^{2}C_{13} + C^{2}C_{23} \\
\tilde{C}_{46} &= S^{2}C^{2}C_{11} - 2S^{2}C^{2}C_{12} + S^{2}C^{2}C_{22} + 6C_{66}C^{4} - 2S^{2}C^{2}C_{66} + C_{66}C^{4} \\
\tilde{C}_{36} &= SC^{2}C_{13} - SC_{23} \\
\tilde{C}_{55} &= C^{2}C_{55} + S^{2}C_{44} \\
\tilde{C}_{45} &= SC^{2}C_{55} - SC_{44} \\
\tilde{C}_{44} &= SC^{2}C_{55} + 2C_{44} \\
\tilde{C}_{33} &= C^{2}C_{33}
\end{align*}
\]
(63)

The definitions \( C = \cos \vartheta \) and \( S = \sin \vartheta \) were used, where \( \vartheta \) is the rotation angle between the material coordinates and problem coordinates.\(^{35}\) If \( \vartheta = 0/90 \) then \( \tilde{C}_{16} = \tilde{C}_{26} = \tilde{C}_{36} = \tilde{C}_{45} = 0 \). The coefficients of Hooke’s law in material coordinates are:
\[
\begin{align*}
\bar{C}_{11} &= 1 - \nu_{12} \nu_{23} \nu_{13} E_{11}; \\
\bar{C}_{12} &= \nu_{21}^{2} + \nu_{23} \nu_{13} E_{12}; \\
\bar{C}_{13} &= \nu_{21} \nu_{13} E_{13}; \\
\bar{C}_{14} &= G_{23}; \\
\bar{C}_{55} &= G_{15}; \\
\bar{C}_{66} &= G_{16} \\
\Delta &= 1 - \nu_{21} \nu_{13} - \nu_{12} \nu_{21} - \nu_{13} \nu_{12} - 2 \nu_{12} \nu_{21} \nu_{13} \\
\nu_{32} &= \frac{E_{12}}{E_{55}} \nu_{23}; \\
\nu_{21} &= \frac{E_{21}}{E_{55}} \nu_{13}; \\
\nu_{31} &= \frac{E_{31}}{E_{55}} \nu_{13}
\end{align*}
\]
(64)

The independent parameters used in the definition of the material properties are nine: \( v_{12}, v_{13}, v_{23}, G_{12}, G_{13}, G_{23}, E_{11}, E_{22} \) and \( E_{33} \). In the case of isotropic materials only two parameters are needed: the Poisson’s ratio and the elastic modulus.

VII. Results

The multilayered structure is a sandwich plate (see Figure 11) made of two skins and a core \( [h_{\text{lower skin}} = h/10; h_{\text{upper skin}} = 2h/10; h_{\text{core}} = (7/10)h] \). It is also \( E_{\text{lower skin}} / E_{\text{upper skin}} = 5/4 \). The plate is simply supported and the load is a sinusoidal pressure applied at the top surface of the plate \( (m = n = 1) \). Different cases are proposed here:

- **Face-to-Core Stiffness Ratio** = \( F_CSR = \frac{E_{\text{upper skin}}}{E_{\text{core}}} \) = 10\(^3\); \( a/h = 4, 100 \)

- **Face-to-Core Stiffness Ratio** = \( F_CSR = \frac{E_{\text{upper skin}}}{E_{\text{core}}} \) = 10\(^5\); \( a/h = 4, 100 \)

As far as Poisson’s ratio is concerned, the following values are used: \( \nu_{\text{lower skin}} = \nu_{\text{upper skin}} = \nu_{\text{core}} = \nu = 0.34 \). In all cases \( b = 3a \). In this test case there is no symmetry with respect the plane \( z = 0 \). The following non-dimensional quantities are introduced:

\[
\begin{align*}
\tilde{u}_{x} &= u_{x} \frac{E_{\text{core}}}{\nu_{\text{core}} h(\frac{z}{h})}; \\
\tilde{u}_{y} &= u_{y} \frac{E_{\text{core}}}{\nu_{\text{core}} h(\frac{z}{h})}; \\
\tilde{u}_{z} &= u_{z} \frac{100 E_{\text{core}}}{\nu_{\text{core}} h(\frac{z}{h})}; \\
\tilde{\sigma}_{xx} &= \frac{\sigma_{xx}}{P_{x}(\frac{z}{h})}; \\
\tilde{\sigma}_{xy} &= \frac{\sigma_{xy}}{P_{x}(\frac{z}{h})}; \\
\tilde{\sigma}_{yy} &= \frac{\sigma_{yy}}{P_{x}(\frac{z}{h})}; \\
\tilde{\sigma}_{zz} &= \frac{\sigma_{zz}}{P_{x}(\frac{z}{h})}; \\
\tilde{\sigma}_{yx} &= \frac{\sigma_{yx}}{P_{x}(\frac{z}{h})}; \\
\tilde{\sigma}_{yy} &= \frac{\sigma_{yy}}{P_{x}(\frac{z}{h})}; \\
\tilde{\sigma}_{xy} &= \frac{\sigma_{xy}}{P_{x}(\frac{z}{h})}; \\
\end{align*}
\]
(65)
All the results have been compared with the solution obtained by solving the “exact” problem. The exact value is indicated with the terminology “elasticity” and is the reference value corresponding to the solution of the differential equations that govern the problem. The details of this elasticity solution are here omitted for brevity.

Tables 1 and 2 compare a large number of ALWT, AHSDT, and AHSHTZ. The orders of the expansions along the thickness is changed. FCSR is dramatically changed to clearly show the limitations of some of the possible approaches. From Tables 1 and 2 it is deduced that when FCSR is very high most of the the Advanced Higher-order Shear Deformation Theories fail to predict the correct displacements or stresses. For example, when $FCSR = 10^5$ even the AHSDT with order 7 of all the displacements ($ED_{777}$) has a large error. The Zig Zag theories present a better compromise (see Figure 17) but only the Layerwise Theories can capture this complex behavior well. The Generalized Unified Formulation allows to compare different types of theories and different types of expansions without any effort. It can be then considered a natural tool to study the numerical performances of multilayered structures in general (see for example reference [46]). The “sensitivity” with respect to the type of theory and/or orders of expansion is an impractical task with the classical approaches but this does not present problems when the Generalized Unified Formulation is used.
<table>
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<th>$a/h$</th>
<th>4</th>
<th>100</th>
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<td>1.51021</td>
</tr>
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<td>3.00603</td>
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<td>1.51021</td>
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</tr>
<tr>
<td>$EDZ_{777}$</td>
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<td>1.51019</td>
</tr>
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Table 1. Comparison of various theories to evaluate the transverse displacements amplitude (center plate deflection) $\hat{u}_z = u_z \frac{100E_{\text{upper skin}}}{\rho \pi h} \text{ in } z = \frac{3}{4} \text{bottom, } x = a/2, y = b/2.$
<table>
<thead>
<tr>
<th>$a/h$</th>
<th>$100$</th>
<th>Err.</th>
<th>$100$</th>
<th>Err.</th>
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</thead>
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<td>$FCSR = 10^5$</td>
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<td>$(0.00)$</td>
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<td>$(0.00)$</td>
<td>$0.33176$</td>
<td>$(0.00)$</td>
</tr>
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<td>$(+3.14)$</td>
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</tr>
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</tr>
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<tr>
<td>$ED_{111}$</td>
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</tr>
</tbody>
</table>

Table 2. Comparison of various theories to evaluate the transverse shear stress $\tilde{\sigma}_{z x} = \frac{\sigma_{z x}}{\gamma} \ln z = \frac{\sigma_{z x}}{\gamma} \ln z_{\text{upper skin}} - \ln \frac{3}{10} h$, $x = 0$, $y = b/2$. The indefinite equilibrium equations have been integrated along the thickness.
Figure 17. Dimensionless in-plane displacement \( \hat{u}_x \).

\[
FCSR = \frac{E_{\text{lower skin}}}{E_{\text{core}}} = 10^5
\]

Parameter values:
- \( m = n = 1 \)
- \( \frac{a}{h} = 4 \)
- \( b = 3a \)

\( \zeta = \frac{z}{h} \)
VIII. Conclusions

An invariant methodology, the Generalized Unified Formulation, to include practically any type of axiomatic theory has been presented. $\infty^3$ Higher-order Shear Deformation Theories, $\infty^3$ Zig-Zag Theories and $\infty^3$ Layer-Wise Theories, with any combination of orders of expansion for the displacement variables can be derived from six invariant $1 \times 1$ matrices. This property gives the possibility to include in a single software and FEM formulation classical and advanced models. It is then possible to find the best formulation, among the infinite possible choices, for the problem under investigation. The applicability of the Generalized Unified Formulation is not confined to the displacement-based case. Mixed variational statements can in fact be used and multfield loadings could be included as well. In any case all the theories are derived from $1 \times 1$ invariant matrices (the so called kernels of the Generalized Unified Formulation). The number of independent kernels depend on the variational statement that is used. In the case of displacement-based theories only six kernels are required, whereas in the case of mixed variational statements this number is different.

The possibility to select the type of theory among the infinite possibilities without actually change the software makes the Generalized Unified Formulation particularly indicated for the solution of optimization and probabilistic problems and for the quasi-3D analysis of structures when localized effects are important. The Generalized Unified Formulation can also have an important educational feature because it allows the user to adopt practically all the existing approaches and to compare with an infinite number of more advanced models. The engineer has only to learn a single invariant formulation and software and this can be applied to study a very large variety of new cases. When the CPU time is more important then a low order theory can be selected and when the accuracy is the determinant factor the order can be increased or the theory can be changed for a more advanced analysis.

The Generalized Unified Formulation is also a powerful tool to evaluate the sensitivity with respect the order of expansion or the type of theory (e.g., Higher-Order Shear Deformation Theories with or without Zig-Zag effects included in the model). This property makes possible to tailor the software to the peculiarity of a new problem with unknown requirements in terms of axiomatic theories necessary to study it. An “intelligent” FEM software can be designed using the Generalized Unified Formulation: the software can change the orders or the theory without the actual need of a new code or theoretical development.

The paper presented an application of these concepts to a sandwich structure. It is demonstrated that when the Face to Core Stiffness Ratio is very high the classical approaches fail and a layerwise formulation is required. This finding is immediate with the Generalized Unified Formulation because all the possible combinations of orders are allowed and also different types of theories can be experimented as well. It is concluded that the Generalized Unified Formulation is a modern approach for the theoretical and computational analysis of multilayered structures.

Acknowledgements

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