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Power allocation in wireless multiuser multi-relay networks with distributed beamforming

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Abstract: This article studies optimal power allocation schemes in a multi-relay cooperating network employing amplify-and-forward (AF) protocol with multiple source–destination pairs. It is assumed that full channel state information is available at the relays. As such, distributed beamforming is employed in forwarding signals to the destinations. In this context, the authors extend recent works on distributed beamforming for a single source–destination pair to the scenario where multiple source–destination pairs are competing for the power resource at the relays. Under orthogonal transmissions of each source–destination pair, considered are the two following power allocation problems: (i) minimise the sum relay power with guaranteed quality of service (QoS) in terms of signal-to-noise ratio (SNR) at the destinations, and (ii) jointly maximise the SNR margin at the destinations subject to individual power constraints at the relays. Although these optimisation problems can be formulated as second-order conic programs (SOCP), the main contribution of this work are proposals of simple and fast converging numerical algorithms, based on the fixed point iteration framework, to efficiently solve these two problems.

1 Introduction

The last few years have witnessed a rapid expansion of wireless communications with a growing number of broadband wireless devices being deployed. In addition, the emergence of the so-called 'Internet of Things' would considerably aggregate the number of wireless networked devices. This trend is expected to put a significant pressure to the current and future wireless network infrastructure to cope with demands for higher throughput, higher robustness and better coverage. On the other hand, the resources that are crucial for wireless communications, namely power and bandwidth, are strictly limited. As a result, meeting these demands would certainly pose many technical challenges to next generation wireless networks.

It is well known that communications over wireless channels usually suffer from poor coverage and low robustness because of the random nature of the wireless medium, which is rich of scattering and susceptible to fadings. Recently, the concept of cooperative communication [1] has been shown as a promising technique that can significantly improve the capacity, reliability and coverage of wireless networks. By deploying a network of cooperating users (objects), one can allow some users to cooperatively act as relays and assist a source user in sending its information symbols to a destination. Since the relays cooperatively form a virtual array of transmit antennas, and thereby providing diversity transmission to the source signals, it widely known that such cooperation can significantly improve the transmission reliability [1]. When these relays know both the backward, that is, source-to-relay ($S \rightarrow R$) and forward, that is, relay-to-destination ($R \rightarrow D$) channels, they can beam their retransmitted signals such that the received signals at the destination are coherently constructed. This cooperative strategy, referred to as distributed beamforming, was investigated in [2–6]. In particular, Jing and Jafarkhani [2] considered the problem of controlling the power resource at each relay in order to maximise the signal-to-noise ratio (SNR) at the destination. It shows that, depending on its own bidirectional channels and other relays' channels, each relay may not transmit at its maximum power to achieve the maximum SNR. Jing and Jafarkhani [2] also provide the condition to determine the optimal transmit power at each relay. The same problem is also considered in [4] by the technique of conic programming. Khajehnouri and Sayed [3] studied a distributed relay strategy for wireless sensor networks to obtain a certain target SNR at the destination, whereas Quek *et al.* [5] investigated a similar problem with the objective of minimising the sum of relay powers, referred to as 'sum relay power' hereafter. More recently, distributed beamforming with second-order statistics of the channel state information was examined in [6].

It is noted that most of the early works in distributed beamforming consider the system with one source and one destination, where all the power consumed at the relays is devoted to only one source–destination pair (each source–destination pair is called a user hereafter). Multiuser multi-relay systems were first investigated in [7, 8], where the relay strategies were proposed to minimise the mean-square error between the source and received signals at the destination. In addition, such systems allow the relays to share their received signals from multiple sources, and thus require reliable links between the relays. In this work, those additional links are not needed as the relays do not share their received signals.

Cooperative multiuser beamforming in wireless *ad hoc* networks was considered in [9], where a cluster of sources cooperatively form beamformers towards multiple destinations. In particular, Li and Wang [9] studied the optimal power allocation and beamforming weights at the sources, as well as provided an efficient iterative algorithm to jointly optimise them. On the other hand, distributed beamforming in a multiuser system was studied in [10–12], where multiple pairs of source and destination were simultaneously assisted by multiple relays on the same channel, that is, over the same frequency band and at the same time. The optimisation problem was formulated as a non-convex quadratically constrained quadratic program. Through convex relaxation technique, such a problem can be efficiently solved by semi-definite programming [10] or by second-order conic programming (SOCP) [11]. Although the approach in [10] is appealing in terms of spectral efficiency (which allow a higher network throughput than that with the orthogonal transmission), the received signals at the destinations suffer from a very high level of interference. This is because the relays do not share their received signals and cannot cooperatively suppress the interference accumulated in the source–relay transmission stage, which then propagates to the destination. To obtain a high signal-to-noise-plus-interference target at the destinations, numerical simulation shows that the system usually requires many relays (order of tens) for a better chance of selecting good relays, which can reduce interference in the forwarding stage [12]. However, deploying many relays is costly in practice and might not be suitable in certain applications. When there is only a small number of relays in the system, it is more preferable to employ orthogonal transmissions for each source–destination pair to completely avoid the inter-user interference at the destinations. At the other extreme, the multiuser multi-relay system model in [13] assumes orthogonality for each relay–destination transmission such that the maximal-ratio combining is possible at the destination. Such an approach is, however, very spectrally inefficient.

In this work, we consider orthogonal transmissions for each source–destination pair in addition to distributed beamforming at the relays. This orthogonality consideration readily applies in practical systems where several single-antenna source–destination pairs operate in orthogonal channels. Examples of such systems are time-division multiple access (TDMA) and orthogonal frequency-division multiple access (OFDMA) systems. In order to improve the coverage and signal reception at the destinations, one could deploy multiple relays to concurrently assist the transmission between each pair. Although the pairs are operating in their own channels, they are actually competing for the coupled power constraint at the relays. Consequently, efficient power utilisation for this network of cooperating objects is critical for both maintaining the source–destination connections and prolonging the power resources at the relays.

The focus of this paper is to study the optimal power allocation schemes at the relays in a multiuser network employing distributed beamforming with the amplify-and-forward (AF) protocol. By means of conic programming, we investigate two main power allocation schemes: (i) minimise the sum relay power with guaranteed quality of service (QoS) in terms of SNR at the destinations, and (ii) jointly maximise the SNR margin at the destinations subject to power constraints at the relays. These problems are sequentially investigated and shown to be closely related with each other.

Under scheme (i), considered are optimisation problems with and without per-relay power constraints. As the two optimisation problems are shown to be second-order conic programs, they can be solved effectively by any conic software package. However, as the required conic package is not always readily available, the approach may not be suitable in real-time communications. To overcome this difficulty, this paper considers an alternative approach by applying the fixed point iteration framework to the relay network, and proposing two simple and fast numerical algorithms to solve the two problems directly. In addition, the proposed algorithms can be implemented in a distributed manner, which allows a decentralised operation in practical networks.

Under scheme (ii), we study two optimal power allocation problems corresponding to two different types of power constraints: sum relay power constraint and per-relay power constraints. Although the two problems can be effectively solved by the bisection method via SOCP feasibility problem, we also propose two simple and fast converging algorithms to directly solve the two problems without the need of a standard conic solution package.

Notations: Superscripts $(\cdot)^{\mathcal{T}}$, $(\cdot)^{*}$, and $(\cdot)^{\mathcal{H}}$ stand for transpose, complex conjugate and complex conjugate transpose operations, respectively; $\mathrm{diag}(d_1, d_2, \ldots, d_M)$ denotes an $M \times M$ diagonal matrix with diagonal elements $d_1, d_2, \ldots, d_M$; $\mathrm{tr}(\cdot)$ denotes the trace of a square matrix; $x^{\star}$ indicates the optimal value of the variable x; $\mathcal{CN}(0, \sigma^2)$ denotes a circularly symmetric Gaussian random variable with variance σ^2.

2 System model

Consider an R-relay network with N pairs of source–destination users (S_n–D_n, $n = 1, \ldots, N$), as illustrated in Fig. 1. (This system model is also applicable to a one-source one-destination OFDM system or a multiple-source multi-destination OFDMA system where N is interpreted as the number of subcarriers.) All relays are assumed to work in a half-duplex mode, that is, they cannot receive and

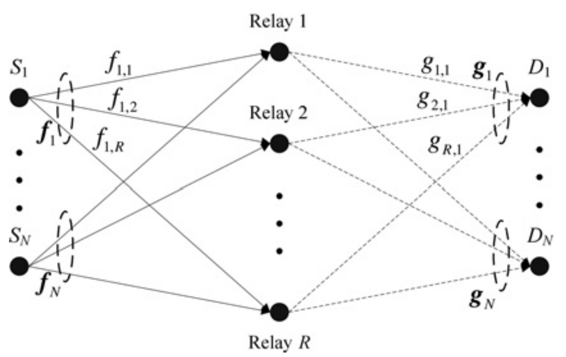

Fig. 1 *Block diagram of a distributed beamforming system with R relays and N users*

transmit at the same time. Assume that there is no direct link between any source and destination and the communication between the two terminals of each user is assisted by all the relays, and implemented in two transmission stages. In the first stage, each user's source broadcasts its signals to all the relays, using TDMA or frequency-division multiple access FDMA. For the nth user, given s_n as the source signal, the received signals at the relays are given by

$$\boldsymbol{r}_n = \boldsymbol{f}_n s_n + \boldsymbol{z}_{r_n} \in \mathbb{C}^{R\times 1} \tag{1}$$

where $\boldsymbol{f}_n = [f_{n,1}, \ldots, f_{n,R}]^{\mathcal{T}}$, and $f_{n,i}$ is the channel from the nth source to the ith relay; $\boldsymbol{z}_{r_n}$ represents the additive white Gaussian noise (AWGN) at the relays, whose components are i.i.d. $\mathcal{CN}(0, \sigma_R^2)$ random variables.

At the ith relay, the received signal for the nth user is amplified by a complex beamforming weight $w_{n,i}$, which is to be designed. Let $\boldsymbol{w}_n = [w_{n,1}, \ldots, w_{n,R}]^{\mathcal{T}}$ be the vector of the beamforming weights for the nth user. Also define $\boldsymbol{W}_n = \mathrm{diag}(\boldsymbol{w}_n)$. Accordingly, by applying the AF protocol [1], the retransmitted signals from the relays scheduled for the nth user are formed as

$$\boldsymbol{t}_n = \boldsymbol{W}_n \boldsymbol{r}_n = \boldsymbol{W}_n \boldsymbol{f}_n s_n + \boldsymbol{W}_n \boldsymbol{z}_{r_n} \tag{2}$$

In the second stage of transmission, all the relays simultaneously transmit to the nth user's destination. Similar to the first stage, the transmission to each user's destination is carried out over orthogonal channels to avoid inter-user interference. Let $\boldsymbol{g}_n = [g_{1,n}, \ldots, g_{R,n}]^{\mathcal{T}}$ represent the channels from R relays to the nth destination. The received signal at the nth destination is written as

$$y_n = \boldsymbol{g}_n^{\mathcal{T}} \boldsymbol{t}_n + z_{d_n} = \boldsymbol{g}_n^{\mathcal{T}} \boldsymbol{W}_n \boldsymbol{f}_n s_n + \boldsymbol{g}_n^{\mathcal{T}} \boldsymbol{W}_n \boldsymbol{z}_{r_n} + z_{d_n} \tag{3}$$

where $z_{d_n} \sim \mathcal{CN}(0, \sigma_D^2)$ is the AWGN at the destination. Define $\boldsymbol{h}_n^* = [h_{n,1}^*, \ldots, h_{n,R}^*]^{\mathcal{T}} = \boldsymbol{f}_n \odot \boldsymbol{g}_n = [f_{n,1}g_{1,n}, \ldots, f_{n,R}g_{R,n}]^{\mathcal{T}}$, where $\odot$ represents the component-wise Hadamard product. As a result, $\boldsymbol{h}_n^*$ models the effective channel from source-n to destination-n through all the relays, excluding the beamforming factors. Then, one has $\boldsymbol{g}_n^{\mathcal{T}} \boldsymbol{W}_n \boldsymbol{f}_n = \boldsymbol{h}_n^{\mathcal{H}} \boldsymbol{w}_n$. Let $\sigma_{S_n}^2 = \mathbb{E}[|s_n|^2]$ be the average transmit power of the nth source. Then, the SNR at the nth destination is given by

$$\mathrm{SNR}_n = \frac{\sigma_{S_n}^2 |\boldsymbol{h}_n^{\mathcal{H}} \boldsymbol{w}_n|^2}{\sigma_R^2 \|\boldsymbol{G}_n^{1/2} \boldsymbol{w}_n\|^2 + \sigma_D^2} \tag{4}$$

where $\boldsymbol{G}_n = \mathrm{diag}(|g_{1,n}|^2, \ldots, |g_{R,n}|^2)$.

Let p_n be the total relay power allocated for the nth user, then p_n is calculated as

$$p_n = \mathbb{E}[\|\boldsymbol{t}_n\|^2] = \boldsymbol{w}_n^{\mathcal{H}} \boldsymbol{D}_n \boldsymbol{w}_n \tag{5}$$

where $\boldsymbol{D}_n$ is an $R \times R$ diagonal matrix, with the ith diagonal element $[\boldsymbol{D}_n]_{ii} = \sigma_{S_n}^2 |f_{n,i}|^2 + \sigma_R^2$. On the other hand, the transmit power at the ith relay is

$$P_i = \sum_{n=1}^{N} \mathbb{E}[|t_{n,i}|^2] = \sum_{n=1}^{N} \boldsymbol{w}_n^{\mathcal{H}} \boldsymbol{D}_n \boldsymbol{E}_i \boldsymbol{w}_n \tag{6}$$

where $\boldsymbol{E}_i$ is an $R \times R$ matrix whose elements are zero, except the (i, i)-element, which is $[\boldsymbol{E}_i]_{ii} = 1$. The total transmit power of all the relays (and for all the users) is given by

$$P_{\text{relay}} = \sum_{i=1}^{R} P_i = \sum_{n=1}^{N} p_n = \sum_{n=1}^{N} \boldsymbol{w}_n^{\mathcal{H}} \boldsymbol{D}_n \boldsymbol{w}_n \tag{7}$$

In this proposed system model, it is assumed that the sources and the relays do not share a common power pool. This assumption arises from practical implementation that the sources are separate entities and usually do not co-locate with the relays. In addition, each source has its own power limit and also does not share its power to other sources. Our interests in this work are to investigate how relays allocate the power between themselves to optimise different system utilities, namely (i) power minimisation with guaranteed QoS or (ii) joint SNR maximisation with power constraints at the relays.

3 Optimal power allocation with guaranteed QoS

3.1 Sum power minimisation without per-relay power constraints

This section considers the optimal power allocation at the relays to minimise the sum relay power given a set of target SNRs at the destinations. This design provides a relaying strategy that can flexibly meet the QoS requirement at each user's destination. The optimisation problem is formulated as follows

$$\begin{aligned} &\underset{\boldsymbol{w}_1,\ldots,\boldsymbol{w}_N}{\text{minimise}} \ \sum_{n=1}^{N} \boldsymbol{w}_n^{\mathcal{H}} \boldsymbol{D}_n \boldsymbol{w}_n \\ &\text{subject to } \mathrm{SNR}_n \geq \gamma_n, \quad \forall n \end{aligned} \tag{8}$$

where γ_n is the target SNR at the nth destination. Obviously, this optimisation problem can be performed separately through N smaller optimisation problems, each corresponds to one user. That is

$$\begin{aligned} &\underset{\boldsymbol{w}_n}{\text{minimise}} \ \boldsymbol{w}_n^{\mathcal{H}} \boldsymbol{D}_n \boldsymbol{w}_n \\ &\text{subject to } \frac{\sigma_{S_n}^2 |\boldsymbol{h}_n^{\mathcal{H}} \boldsymbol{w}_n|^2}{\sigma_R^2 \|\boldsymbol{G}_n^{1/2} \boldsymbol{w}_n\|^2 + \sigma_D^2} \geq \gamma_n \end{aligned} \tag{9}$$

There are several approaches to solve the above optimisation problem. The first approach, first proposed in [6], is to establish $\boldsymbol{w}_n$ in the direction of $\boldsymbol{D}_n^{-1/2} \boldsymbol{x}_n$, where $\boldsymbol{x}_n$ is the the principal eigenvector of $\boldsymbol{D}_n^{-1/2} (\sigma_{S_n}^2 \boldsymbol{h}_n \boldsymbol{h}_n^{\mathcal{H}} - \gamma_n \sigma_R^2 \boldsymbol{G}_n) \boldsymbol{D}_n^{-1/2}$. Another approach is to cast the SNR constraint as a second-order conic constraint [5]

$$\sqrt{\frac{\sigma_{S_n}^2}{\gamma_n}} \boldsymbol{h}_n^{\mathcal{H}} \boldsymbol{w}_n \geq \left\| \begin{matrix} \sigma_R \boldsymbol{G}_n^{1/2} \boldsymbol{w}_n \\ \sigma_D \end{matrix} \right\| \tag{10}$$

and then solve the optimisation problem as an SOCP. The solution to the problem can be obtained from any standard conic solution package, such as cvx [14]. We now consider an alternative approach to solve problem (9) by optimising the relay power factor p_n directly, instead of dealing with the beamforming vector. The new approach, which does not rely on external conic solution software, also motivates a

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simple iterative fixed point algorithm. Besides, we note that the proposed algorithm is useful in solving the sum relay power minimisation with per-relay power constraints in Section 3.2 and the inverse problems that maximise the SNR margin in Section 4.

First, problem (9) can also be recast as

$$\begin{aligned} &\underset{\boldsymbol{w}_n, p_n}{\text{minimise}} \ p_n \\ &\text{subject to} \ \frac{\sigma_{S_n}^2 |\boldsymbol{h}_n^{\mathcal{H}} \boldsymbol{w}_n|^2}{\sigma_R^2 \|\boldsymbol{G}_n^{1/2} \boldsymbol{w}_n\|^2 + \sigma_D^2} \geq \gamma_n \\ &\qquad \boldsymbol{w}_n^{\mathcal{H}} \boldsymbol{D}_n \boldsymbol{w}_n = p_n \end{aligned} \tag{11}$$

Second, the following lemma establishes the relation between the optimal beamforming vector $\boldsymbol{w}_n$ and the allocated relay power p_n.

Lemma 1: Given p_n as the relay power allocated for user-n, the optimal beamforming weights at the relays to maximise the SNR of user-n are

$$w_{n,i} = \frac{\sqrt{\delta_n} f_{n,i}^{\star} g_{i,n}^{\star}}{p_n \sigma_R^2 |g_{i,n}|^2 + \sigma_D^2 (\sigma_{S_n}^2 |f_{n,i}|^2 + \sigma_R^2)} \tag{12}$$

where the normalisation factor δ_n is

$$\delta_n = \frac{p_n}{\sum_{i=1}^{R} (|f_{n,i}|^2 |g_{i,n}|^2 (\sigma_{S_n}^2 |f_{n,i}|^2 + \sigma_R^2) / [p_n \sigma_R^2 |g_{i,n}|^2 + \sigma_D^2 (\sigma_{S_n}^2 |f_{n,i}|^2 + \sigma_R^2)]^2)} \tag{13}$$

The corresponding maximum SNR is

$$\mathrm{SNR}_n(p_n) = \sum_{i=1}^{R} \frac{p_n \sigma_{S_n}^2 |f_{n,i}|^2 |g_{i,n}|^2}{p_n \sigma_R^2 |g_{i,n}|^2 + \sigma_D^2 (\sigma_{S_n}^2 |f_{n,i}|^2 + \sigma_R^2)} \tag{14}$$

Proof: The derivation of the optimal beamforming weight vector $\boldsymbol{w}_n$ is followed from the Rayleigh–Ritz theorem [15] and Proposition 1 of [16]. Substituting $\sigma_D^2 = (\sigma_D^2 \boldsymbol{w}_n^{\mathcal{H}} \boldsymbol{D}_n \boldsymbol{w}_n / p_n)$ into the SNR expression in (4) gives

$$\begin{aligned} \mathrm{SNR}_n &= \frac{\sigma_{S_n}^2 \boldsymbol{w}_n^{\mathcal{H}} \boldsymbol{h}_n \boldsymbol{h}_n^{\mathcal{H}} \boldsymbol{w}_n}{\boldsymbol{w}_n^{\mathcal{H}} (\sigma_R^2 \boldsymbol{G}_n + (\sigma_D^2 / p_n) \boldsymbol{D}_n) \boldsymbol{w}_n} \\ &= \frac{p_n \sigma_{S_n}^2 \boldsymbol{w}_n^{\mathcal{H}} \boldsymbol{h}_n \boldsymbol{h}_n^{\mathcal{H}} \boldsymbol{w}_n}{\boldsymbol{w}_n^{\mathcal{H}} (p_n \sigma_R^2 \boldsymbol{G}_n + \sigma_D^2 \boldsymbol{D}_n) \boldsymbol{w}_n} \leq p_n \sigma_{S_n}^2 \lambda_{\max} \end{aligned}$$

where $\lambda_{\max}$ is the largest eigenvalue of $\boldsymbol{B}_n^{-1/2} \boldsymbol{h}_n \boldsymbol{h}_n^{\mathcal{H}} (\boldsymbol{B}_n^{\mathcal{H}})^{-1/2}$, with $\boldsymbol{B}_n = p_n \sigma_R^2 \boldsymbol{G}_n + \sigma_D^2 \boldsymbol{D}_n$. The equality holds if $\boldsymbol{w}_n \propto \boldsymbol{B}_n^{-1} \boldsymbol{h}_n$. We note that this result is consistent with the work in [17] on the generalised Rayleigh–Ritz theorem. More specifically, (13) of [17] states that $\boldsymbol{w}_n$ is a scaled version of the principal eigenvector of $\boldsymbol{B}_n^{-1} \boldsymbol{h}_n \boldsymbol{h}_n^{\mathcal{H}}$, which is indeed $\boldsymbol{B}_n^{-1} \boldsymbol{h}_n$.

We proceed to provide the closed-form expression of each beamforming weight in (12), and the normalisation factor δ_n to ensure $\boldsymbol{w}_n^{\mathcal{H}} \boldsymbol{D}_n \boldsymbol{w}_n = p_n$. It is also noted that the largest eigenvalue of $\boldsymbol{B}_n^{-1/2} \boldsymbol{h}_n \boldsymbol{h}_n^{\mathcal{H}} (\boldsymbol{B}_n^{\mathcal{H}})^{-1/2}$ is its only non-zero eigenvalue, which is also its trace. Thus, the obtained optimal SNR value can be found in a closed-form expression as stated in (14). □

By applying Lemma 1, one can optimise the power allocation p_n for user-n, then determine the optimal beamforming vector accordingly. For notational simplicity, let

$$a_{n,i} = \frac{\sigma_{S_n}^2}{\sigma_R^2} |f_{n,i}|^2, \quad b_{n,i} = \frac{\sigma_D^2 (\sigma_{S_n}^2 |f_{n,i}|^2 + \sigma_R^2)}{\sigma_R^2 |g_{i,n}|^2}$$

Then, the achievable SNR can be written as

$$\mathrm{SNR}_n(p_n) = \sum_{i=1}^{R} \frac{a_{n,i}\, p_n}{b_{n,i} + p_n} \tag{15}$$

Interestingly, $\mathrm{SNR}_n(p_n)$ is a concave increasing function in p_n.

The optimisation problem is now restated as

$$\begin{aligned} &\underset{p_n}{\text{minimise}} \ p_n \\ &\text{subject to} \ \sum_{i=1}^{R} \frac{a_{n,i} p_n}{b_{n,i} + p_n} \geq \gamma_n \end{aligned} \tag{16}$$

The above problem is convex, which then can be solved efficiently by standard convex optimisation algorithms. In addition, the structure of the restated problem also reveals several interesting properties of the problem, including its feasibility and solution. Since $p_n/(b_{n,i} + p_n) < 1$, one has

$$\mathrm{SNR}_n = \sum_{i=1}^{R} \frac{a_{n,i}\, p_n}{b_{n,i} + p_n} < \sum_{i=1}^{R} a_{n,i}$$

Thus, if the target SNR $\gamma_n \geq \sum_{i=1}^{R} a_{n,i}$, the problem will be infeasible.

Now, suppose that the target SNR is set such that the problem is feasible. Since $\sum_{i=1}^{R} (a_{n,i} p_n / b_{n,i} + p_n)$ is a monotonically increasing function, the constraint $\sum_{i=1}^{R} (a_{n,\,i}\, p_n / b_{n,i} + p_n) \geq \gamma_n$ must be met with equality at optimum. Thus, the unique solution of

$$\sum_{i=1}^{R} \frac{a_{n,\,i}\, p_n}{b_{n,i} + p_n} = \gamma_n \tag{17}$$

is also the optimal solution to (16). It is then of interest to find a simple and fast numerical algorithm to solve the Rth polynomial in (17). The monotonicity of $\sum_{i=1}^{R} (a_{n,\,i} p_n / b_{n,i} + p_n)$ makes the bisection method especially suitable to find the solution. Note that the structure in (17) also motivates a simple iterative fixed point algorithm to find the optimal $p_n^{\star}$. By rearranging (17), one has the following simple iteration

$$p_n^{(t+1)} = \frac{\gamma_n}{\sum_{i=1}^{R} (a_{n,i} / b_{n,i} + p_n^{(t)})} \tag{18}$$

If (17) is feasible, then the above iteration will converge from any initial point $p_n^{(0)} \geq 0$. The convergence analysis of the fixed point iteration is based on the standard function approach introduced in [18]. Denote $f_n(p_n^{(t)}) = \gamma_n / (\sum_{i=1}^{R} (a_{n,i} / b_{n,i} + p_n^{(t)}))$, then the fixed point iteration $p_n^{(t+1)} = f_n(p_n^{(t)})$ will converge to a unique fixed point $p_n^{\star}$ if the function $f_n(p_n)$ obeys the following properties [18]:

1. Positivity: $f_n(p_n) > 0$ for all $p_n > 0$.

2. Monotonicity: if $p_n > p_{n'}$, then $f_n(p_n) > f_n(p_{n'})$.
3. Scalability: if $\alpha > 1$, then $\alpha f_n(p_n) > f_n(\alpha p_n)$.

It is easy to verify that all these three properties are satisfied by the function $f_n(p_n)$. Thus, the fixed point iteration (18) will surely converge if (17) is feasible. Numerical results show that the proposed algorithm converges in a few iterations.

3.2 Sum power minimisation with per-relay power constraints

In the previous section, sum relay power minimisation with guaranteed QoS at the destinations was considered, where no restrictions on the individual power at each relay were imposed. However, in practical relay communications, the relays are not normally co-located. In addition, each relay is equipped with its own amplifier and typically has its own power limit. Under the per relay power constraints, the power allocation scheme has to be modified accordingly while meeting the SNR requirement at each user's receiving end. This section considers the approach to uniformly minimise the margin $P_i/P_i^{\max}$ over all the relays, where $P_i^{\max}$ denotes the maximum transmit power of the ith relay. The approach of minimising the power consumption margin was first investigated for the multiuser beamforming downlink problem in point-to-point communications [19]. When applied to relay networks, the idea is to serve all the users, while maintaining the balance in power consumption at the relays. The problem is stated as follows

$$\begin{aligned} &\underset{\alpha, \boldsymbol{w}_1,\ldots,\boldsymbol{w}_N}{\text{minimise}} \ \alpha \\ &\text{subject to } \mathrm{SNR}_n \ge \gamma_n, \quad \forall n \\ &\qquad P_i \le \alpha P_i^{\max}, \quad \forall i \\ &\qquad \alpha \le 1 \end{aligned} \tag{19}$$

Here, the last two constraints are to ensure the power consumption at each relay not to exceed its maximum allowable amount. As presented before, the SNR constraints are feasible if $\gamma_n \le \sum_{i=1}^{R} a_{n,i}$, $\forall n$. However, the feasibility of the power constraints in (19) is much more difficult to determine analytically. Thus, we drop the last constraint $\alpha \le 1$ from problem (19) and consider the following optimisation

$$\begin{aligned} &\underset{\alpha, \boldsymbol{w}_1,\ldots,\boldsymbol{w}_N}{\text{minimise}} \ \alpha \sum_{i=1}^{R} P_i^{\max} \\ &\text{subject to } \frac{\sigma_{S_n}^2 |\boldsymbol{h}_n^{\mathcal{H}} \boldsymbol{w}_n|^2}{\sigma_R^2 \|\boldsymbol{G}_n^{1/2}\boldsymbol{w}_n\|^2 + \sigma_D^2} \ge \gamma_n, \quad \forall n \\ &\qquad \sum_{n=1}^{N} \boldsymbol{w}_n^{\mathcal{H}} \boldsymbol{D}_n \boldsymbol{E}_i \boldsymbol{w}_n \le \alpha P_i^{\max}, \quad \forall i \end{aligned} \tag{20}$$

which can be interpreted as a sum relay power minimisation problem with per-relay power constraint awareness. The resultant optimal $\alpha^\star$ from problem (20) then numerically determines the feasibility of problem (19). Note that the feasibility of problem (20) only depends on the feasibility of the SNR constraints, since α can be increased to meet the per-relay power constraints. However, it might happen that $\alpha^\star > 1$ at the optimal solution, that is, it is infeasible to find the beamforming vectors that meet both the QoS constraints and the strict per-relay power constraints. To handle this situation, an inverse problem, which tries to maximise the SNR under strict per-relay power constraints $P_i \le P_i^{\max}$, is desirable. We investigate such an inverse problem in Section 4.

It is also noted that the optimisation problem stated in (20) is not readily convex. However, as the SNR constraints can be recast as an SOC constraint as in (10), the problem can be transformed into a convex one, namely

$$\begin{aligned} &\underset{\alpha, \boldsymbol{w}_1,\ldots,\boldsymbol{w}_N}{\text{minimise}} \ \alpha \sum_{i=1}^{R} P_n^{\max} \\ &\text{subject to } \sqrt{\frac{\sigma_{S_n}^2}{\gamma_n}} \boldsymbol{h}_n^{\mathcal{H}} \boldsymbol{w}_n \ge \left\| \begin{matrix} \sigma_R \boldsymbol{G}_n^{1/2} \boldsymbol{w}_n \\ \sigma_D \end{matrix} \right\|, \quad \forall n \\ &\qquad \sum_{n=1}^{N} \boldsymbol{w}_n^{\mathcal{H}} \boldsymbol{D}_n \boldsymbol{E}_i \boldsymbol{w}_n \le \alpha P_i^{\max}, \quad \forall i \end{aligned} \tag{21}$$

Interestingly, the Lagrangians of the non-convex form in (20) and the convex form in (21) are the same. This property can be easily verified by following the similar technique to that in Proposition 1 of [19]. As strong duality holds for a convex problem [20], strong duality also holds for problem (20). This means that the optimal value of problem (20) can be found by its dual problem. In the following, we investigate the dual problem of (20), which then reveals both the structure of the original problem's solution and the algorithm to solve it.

3.2.1 Beamforming duality: The Lagrangian of problem (20) is established as

$$\begin{aligned} &\mathcal{L}(\alpha, \boldsymbol{w}_n, \boldsymbol{\lambda}, \boldsymbol{\mu}) \\ &= \alpha \sum_{i=1}^{R} P_i^{\max} + \sum_{i=1}^{R} \mu_i \left(\sum_{n=1}^{N} \boldsymbol{w}_n^{\mathcal{H}} \boldsymbol{D}_n \boldsymbol{E}_i \boldsymbol{w}_n - \alpha P_i^{\max} \right) \\ &\quad - \sum_{n=1}^{N} \lambda_n \left(\frac{\sigma_{S_n}^2}{\gamma_n} |\boldsymbol{h}_n^{\mathcal{H}} \boldsymbol{w}_n|^2 - \sigma_R^2 \|\boldsymbol{G}_n^{1/2} \boldsymbol{w}_n\|^2 - \sigma_D^2 \right) \end{aligned} \tag{22}$$

Denote $\boldsymbol{Q} = \mathrm{diag}(\mu_1, \ldots, \mu_R)$ and $\boldsymbol{P} = \mathrm{diag}(P_1^{\max}, \ldots, P_R^{\max})$. Rearrange the Lagrangian $\mathcal{L}(\alpha, \boldsymbol{w}_n, \boldsymbol{\lambda}, \boldsymbol{\mu})$ in (22), one has

$$\begin{aligned} \mathcal{L}(\alpha, \boldsymbol{w}_n, \boldsymbol{\lambda}, \boldsymbol{Q}) = &\sum_{n=1}^{N} \lambda_n \sigma_D^2 + \sum_{n=1}^{N} \mathcal{L}_n(\boldsymbol{w}_n, \lambda_n, \boldsymbol{Q}) \\ &- \alpha[\mathrm{tr}(\boldsymbol{QP}) - \mathrm{tr}(\boldsymbol{P})] \end{aligned} \tag{23}$$

where $\mathcal{L}_n(\boldsymbol{w}_n, \lambda_n, \boldsymbol{Q}) = \boldsymbol{w}_n^{\mathcal{H}}(\boldsymbol{D}_n \boldsymbol{Q} - (\lambda_n \sigma_{S_n}^2/\gamma_n) \boldsymbol{h}_n \boldsymbol{h}_n^{\mathcal{H}} + \lambda_n \sigma_R^2 \boldsymbol{G}_n) \boldsymbol{w}_n$ only depends on $\boldsymbol{w}_n$, λ_n and $\boldsymbol{Q}$. The dual function of (23) is established as

$$\begin{aligned} g(\boldsymbol{Q}, \boldsymbol{\lambda}) = &\sum_{n=1}^{N} \lambda_n \sigma_D^2 + \sum_{n=1}^{N} \min_{\boldsymbol{w}_n} \mathcal{L}_n(\boldsymbol{w}_n, \lambda_n, \boldsymbol{Q}) \\ &- \min_{\alpha} \{\alpha[\mathrm{tr}(\boldsymbol{QP}) - \mathrm{tr}(\boldsymbol{P})]\} \end{aligned}$$

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It is clear that if

$$\boldsymbol{D}_n\boldsymbol{Q} - \frac{\lambda_n \sigma_{S_n}^2}{\gamma_n}\boldsymbol{h}_n\boldsymbol{h}_n^{\mathcal{H}} + \lambda_n\sigma_R^2\boldsymbol{G}_n$$

is not a positive semidefinite matrix, there exists $\boldsymbol{w}_n$ to make $\mathcal{L}_n$ unbounded below. Similarly, if $\mathrm{tr}(\boldsymbol{QP})$–$\mathrm{tr}(\boldsymbol{P}) > 0$, it is possible to find $\alpha > 0$ to make $\alpha[\mathrm{tr}(\boldsymbol{QP}) - \mathrm{tr}(\boldsymbol{P})] = -\infty$. Thus, the dual problem is stated as

$$\begin{aligned} &\underset{\boldsymbol{Q}}{\text{maximise}} \ \max_{\boldsymbol{\lambda}} \sum_{n=1}^{N} \lambda_n \sigma_D^2 \\ &\text{subject to } \boldsymbol{D}_n\boldsymbol{Q} + \lambda_n\sigma_R^2\boldsymbol{G}_n \succeq \frac{\lambda_n\sigma_{S_n}^2}{\gamma_n}\boldsymbol{h}_n\boldsymbol{h}_n^{\mathcal{H}}, \quad \forall n \\ &\qquad \mathrm{tr}(\boldsymbol{QP}) \le \mathrm{tr}(\boldsymbol{P}), \quad \boldsymbol{Q} \text{ is diagonal}, \quad \boldsymbol{Q} \succeq \boldsymbol{0} \end{aligned} \tag{24}$$

Since strong duality holds for problem (20), we have

$$\alpha^\star \sum_{i=1}^{R} P_i^{\max} = \sum_{n=1}^{N} \lambda_n^\star \sigma_D^2 \tag{25}$$

In the next section, an interpretation via a virtual single-input multiple-output (SIMO) uplink channel shows that the dual problem (24) is equivalent to the following minimax problem

$$\begin{aligned} &\underset{\boldsymbol{Q}}{\text{maximise}} \ \min_{\boldsymbol{\lambda}, \hat{\boldsymbol{w}}_n} \sum_{n=1}^{N} \lambda_n \sigma_D^2 \\ &\text{subject to } \frac{\lambda_n\sigma_{S_n}^2 |\boldsymbol{h}_n^{\mathcal{H}}\hat{\boldsymbol{w}}_n|^2}{\hat{\boldsymbol{w}}_n^{\mathcal{H}}\boldsymbol{D}_n\boldsymbol{Q}\hat{\boldsymbol{w}}_n + \lambda_n\sigma_R^2\hat{\boldsymbol{w}}_n^{\mathcal{H}}\boldsymbol{G}_n\hat{\boldsymbol{w}}_n} \ge \gamma_n, \quad \forall n \\ &\qquad \mathrm{tr}(\boldsymbol{QP}) \le \mathrm{tr}(\boldsymbol{P}), \quad \boldsymbol{Q} \text{ is diagonal}, \quad \boldsymbol{Q} \succeq \boldsymbol{0} \end{aligned} \tag{26}$$

where $\hat{\boldsymbol{w}}_n^{\mathcal{H}}$ is interpreted as the receive beamforming vector of the virtual uplink channel for user-n.

3.2.2 Interpretation via a virtual uplink channel: In point-to-point multiuser communications, it is widely known that the optimal beamforming design for the downlink multiple-input multiple-output (MIMO) channel can be found via its equivalent uplink channel, which is much easier to handle. This property is known as uplink–downlink duality [19, 21–23]. Inspired by the uplink–downlink duality property of the MIMO channel, here we introduce the concept of a virtual uplink channel, and uses it to find the optimal power allocation scheme in a multi-relay network with per-relay power constraints.

Consider a virtual SIMO uplink channel where a single-antenna transmitter with power $\hat{p}_n$ wants to communicate with an R-antenna receiver. The channel is modelled as $\sigma_{S_n}\boldsymbol{h}_n \in \mathbb{C}^{1\times R}$. The effective additive Gaussian noise at the receiver has the following covariance: $\sigma_D^2\boldsymbol{D}_n\boldsymbol{Q} + \hat{p}_n\sigma_R^2\boldsymbol{G}_n$. One can interpret $\sigma_D^2\boldsymbol{D}_n\boldsymbol{Q}$ as the added noise at the receiver and $\hat{p}_n\sigma_R^2\hat{\boldsymbol{G}}_n$ as the noise induced by the transmitter, which depends on the transmit power $\hat{p}_n$. Then, it is of interest to find the optimal combiner at the receiver and the minimal transmit power $\hat{p}_n$ at the transmitter to obtain a certain target SNR γ_n at the virtual uplink channel's receiving end.

Let $\hat{\boldsymbol{w}}_n^{\mathcal{H}}$ be the receive beamforming vector. The SNR at the receiver can be expressed as

$$\widehat{\mathrm{SNR}}_n = \frac{\hat{p}_n\sigma_{S_n}^2 |\boldsymbol{h}_n^{\mathcal{H}}\hat{\boldsymbol{w}}_n|^2}{\sigma_D^2\hat{\boldsymbol{w}}_n^{\mathcal{H}}\boldsymbol{D}_n\boldsymbol{Q}\hat{\boldsymbol{w}}_n + \hat{p}_n\sigma_R^2\hat{\boldsymbol{w}}_n^{\mathcal{H}}\boldsymbol{G}_n\hat{\boldsymbol{w}}_n} \tag{27}$$

Similar to the proof of Lemma 1, to maximise the above SNR, using the Rayleigh–Ritz theorem [15], the optimal receive beamformer is

$$\hat{\boldsymbol{w}}_n = (\sigma_D^2\boldsymbol{D}_n\boldsymbol{Q} + \hat{p}_n\sigma_R^2\boldsymbol{G}_n)^{-1}\boldsymbol{h}_n \tag{28}$$

Given a specific value of the transmit power $\hat{p}_n$, the weight of the optimal combiner at the ith receive antenna only depends on the channel connected to itself, and is given by

$$\hat{w}_{n,i} = \frac{f_{n,i}^\star g_{i,n}^\star}{(\sigma_{S_n}^2|f_{n,i}|^2 + \sigma_R^2)\sigma_D^2\mu_n + \hat{p}_n\sigma_R^2|g_{i,n}|^2} \tag{29}$$

With the optimal combiner, the constraint on the SNR at the receiver, $\widehat{\mathrm{SNR}}_n \ge \gamma_n$, is now equivalent to

$$\hat{p}_n\sigma_{S_n}^2\boldsymbol{h}_n^{\mathcal{H}}(\sigma_D^2\boldsymbol{D}_n\boldsymbol{Q} + \hat{p}_n\sigma_R^2\boldsymbol{G}_n)^{-1}\boldsymbol{h}_n \ge \gamma_n \tag{30}$$

The next task is to determine the minimal uplink transmit power $\hat{p}_n$. This problem is stated as

$$\begin{aligned} &\underset{\hat{p}_n}{\text{minimise}} \ \hat{p}_n \\ &\text{subject to } \hat{p}_n\sigma_{S_n}^2\boldsymbol{h}_n^{\mathcal{H}}(\sigma_D^2\boldsymbol{D}_n\boldsymbol{Q} + \hat{p}_n\sigma_R^2\boldsymbol{G}_n)^{-1}\boldsymbol{h}_n \ge \gamma_n \end{aligned} \tag{31}$$

If the above inequality is reversed, the optimisation problem can also be reversed into a maximisation problem as follows

$$\begin{aligned} &\underset{\hat{p}_n}{\text{maximise}} \ \hat{p}_n \\ &\text{subject to } \hat{p}_n\sigma_{S_n}^2\boldsymbol{h}_n^{\mathcal{H}}(\sigma_D^2\boldsymbol{D}_n\boldsymbol{Q} + \hat{p}_n\sigma_R^2\boldsymbol{G}_n)^{-1}\boldsymbol{h}_n \le \gamma_n \end{aligned} \tag{32}$$

It should be noted that the reversals of the constraint and the minimisation into a maximisation do not effect the solution of (31), as the two constraints in (31) and in (32) are met with equality at optimality.

Observe that the constraint in (32) is equivalent to

$$\sigma_D^2\boldsymbol{D}_n\boldsymbol{Q} + \hat{p}_n\sigma_R^2\boldsymbol{G}_n \succeq \frac{\hat{p}_n\sigma_{S_n}^2}{\gamma_n}\boldsymbol{h}_n\boldsymbol{h}_n^{\mathcal{H}}$$

(from Lemma 1 in [19]). Furthermore, identify $\hat{p}_n = \lambda_n\sigma_D^2$, then the minimax problem in (26) must be equivalent to the dual problem (24). This allows us to solve the power allocation problem with per-relay power constraints by solving the minimax problem (26).

3.2.3 Numerical algorithm: The minimax problem (26) can be solved by iteratively solving N inner minimisation problems on $(\hat{\boldsymbol{w}}_n, \lambda_n)$ and the outer maximisation problem on $\boldsymbol{Q}$. For the outer problem, we compute the maximisation

$$\begin{aligned} &\underset{\boldsymbol{Q}}{\text{maximise}} \ f(\boldsymbol{Q}) \\ &\text{subject to } \mathrm{tr}(\boldsymbol{QP}) \le \mathrm{tr}(\boldsymbol{P}), \quad \boldsymbol{Q} \text{ is diagonal}, \quad \boldsymbol{Q} \succeq \boldsymbol{0} \end{aligned} \tag{33}$$

where $f(\boldsymbol{Q})$ is defined as

$$f(\boldsymbol{Q}) = \underset{l,\hat{\boldsymbol{w}}_n}{\text{minimise}} \sum_{n=1}^{N} \lambda_n \sigma_D^2$$
$$\text{subject to } \frac{\lambda_n \sigma_{S_n}^2 |\boldsymbol{h}_n^{\mathcal{H}} \hat{\boldsymbol{w}}_n|^2}{\hat{\boldsymbol{w}}_n^{\mathcal{H}} \boldsymbol{D}_n \boldsymbol{Q} \hat{\boldsymbol{w}}_n + \lambda_n \sigma_R^2 \hat{\boldsymbol{w}}_n^{\mathcal{H}} \boldsymbol{G}_n \hat{\boldsymbol{w}}_n} \geq \gamma_n, \quad \forall n$$

For the inner problems, with a fixed $\boldsymbol{Q}$ the optimal combiner $\hat{\boldsymbol{w}}_n$ for the user-n is given by (28), and the optimal power factor λ_n is obtained by solving problem (31). Note that problem (31) is equivalent to

$$\underset{\hat{p}_n}{\text{minimise}} \ \hat{p}_n$$
$$\text{subject to } \sum_{i=1}^{R} \frac{\hat{p}_n a_{n,i}}{b_{n,i}\mu_i + \hat{p}_n} \geq \gamma_n \tag{34}$$

Thus, the optimal value $\hat{p}_n^\star$ can be obtained from the simple fixed point iteration, as presented in Section 3.1. Moreover, with $\lambda_n = \hat{p}_n/\sigma_D^2$, the fixed point iteration

$$\lambda_n^{(t+1)} = \frac{\gamma_n}{\sigma_D^2 \sum_{i-1}^{R} (a_{n,i}/b_{n,i}\mu_i + \sigma_D^2 \lambda_n^{(t)})} \tag{35}$$

will surely converge to the optimal value $\lambda_n^\star$.

Having known the optimal combiner of the virtual uplink channel $\hat{\boldsymbol{w}}_n$ and its power factor $\lambda_n \sigma_D^2$, the optimal distributed beamformer $\boldsymbol{w}_n$ for the user-n can be determined by exploiting the relation between the two beamformers in the next lemma.

Lemma 2: The optimal distributed beamforming vector $\boldsymbol{w}_n$ in the multiuser beamforming problem is a scaled version of $\hat{\boldsymbol{w}}_n$, that is, $\boldsymbol{w}_n = \sqrt{\zeta_n}\hat{\boldsymbol{w}}_n$.

Proof: From the Karush–Kuhn–Tucker conditions [20], the gradient of Lagrangian $\mathcal{L}_n(\boldsymbol{Q}, \lambda_n, \boldsymbol{w}_n)$ vanishes at the optimum of $\boldsymbol{w}_n$, that is

$$\frac{\partial \mathcal{L}_n}{\partial \boldsymbol{w}_n^*} = \left(\boldsymbol{D}_n \boldsymbol{Q} - \frac{\lambda_n \sigma_{S_n}^2}{\gamma_n} \boldsymbol{h}_n \boldsymbol{h}_n^{\mathcal{H}} + \lambda_n \sigma_R^2 \boldsymbol{G}_n \right) \boldsymbol{w}_n = \boldsymbol{0}$$

Thus

$$\boldsymbol{w}_n = (\boldsymbol{D}_n \boldsymbol{Q} + \lambda_n \sigma_R^2 \boldsymbol{G}_n)^{-1} \frac{\lambda_n \sigma_{S_n}^2}{\gamma_n} \boldsymbol{h}_n \boldsymbol{h}_n^{\mathcal{H}} \boldsymbol{w}_n$$
$$= \frac{\lambda_n \sigma_D^2 \sigma_{S_n}^2 \boldsymbol{h}_n^{\mathcal{H}} \boldsymbol{w}_n}{\gamma_n} \hat{\boldsymbol{w}}_n$$

which suggests $\sqrt{\zeta_n} = (\lambda_n \sigma_D^2 \sigma_{S_n}^2/\gamma_n)\boldsymbol{h}_n^{\mathcal{H}} \boldsymbol{w}_n$. However, this expression still shows the dependence of ζ_n on $\boldsymbol{w}_n$. The next step is to determine the value ζ_n based on $\hat{\boldsymbol{w}}_n$. As the SNR constraint in (20) is met with equality at optimum, that is, $(\sigma_{S_n}^2/\gamma_n)|\boldsymbol{h}_n^{\mathcal{H}} \boldsymbol{w}_n|^2 = \sigma_R^2 \boldsymbol{w}_n^{\mathcal{H}} \boldsymbol{G}_n \boldsymbol{w}_n + \sigma_D^2$. Substituting $\boldsymbol{w}_n = \sqrt{\zeta_n}\hat{\boldsymbol{w}}_n$ into the SNR constraint, one has $(\zeta_n \sigma_{S_n}^2/\gamma_n)|\boldsymbol{h}_n^{\mathcal{H}} \hat{\boldsymbol{w}}_n|^2 = \zeta_n \sigma_R^2 \hat{\boldsymbol{w}}_n^{\mathcal{H}} \boldsymbol{G}_n \hat{\boldsymbol{w}}_n + \sigma_D^2$ Therefore

$$\zeta_n = \frac{\gamma_n \sigma_D^2}{\sigma_{S_n}^2 |\boldsymbol{h}_n^{\mathcal{H}} \hat{\boldsymbol{w}}_n|^2 - \gamma_n \sigma_R^2 \hat{\boldsymbol{w}}_n^{\mathcal{H}} \boldsymbol{G}_n \hat{\boldsymbol{w}}_n} \tag{36}$$

We now return to the maximisation problem of $f(\boldsymbol{Q})$ in (33). This problem can be computed by the subgradient projection method, as presented next.

Lemma 3: The function $f(\boldsymbol{Q})$ is concave in $\boldsymbol{Q}$, and its subgradient is given by $\text{diag}(\sum_{n=1}^{N} \boldsymbol{w}_n \boldsymbol{w}_n^{\mathcal{H}} \boldsymbol{D}_n)$, where $\boldsymbol{w}_n$ is the optimal distributed beamforming vector obtained from Lemma 2.

Proof: The proof of this lemma is similar to the proof of Proposition 3 in [19] for the point-to-point multiuser downlink beamforming problem. Since $f(\boldsymbol{Q})$ is the objective function of the dual problem, it is a concave function by nature [20]. Now look back at the Lagrangian of the distributed beamforming problem in (22). For a fixed $\boldsymbol{Q}$, one has

$$f(\boldsymbol{Q}) = \min_{\boldsymbol{w}_1,\ldots,\boldsymbol{w}_n} \min_{\alpha} \mathcal{L}(\alpha, \boldsymbol{w}_n, \boldsymbol{\lambda}, \boldsymbol{Q})$$
$$= \min_{\boldsymbol{w}_1,\ldots,\boldsymbol{w}_n} \sum_{n=1}^{N} \boldsymbol{w}_n^{\mathcal{H}} \boldsymbol{D}_n \boldsymbol{Q} \boldsymbol{w}_n \tag{37}$$
$$\text{subject to } \frac{\sigma_{S_n}^2}{\gamma_n} |\boldsymbol{h}_n^{\mathcal{H}} \boldsymbol{w}_n|^2 \geq \sigma_R^2 \|\boldsymbol{G}_n^{1/2} \boldsymbol{w}_n\|^2 + \sigma_D^2$$

Following the same procedure in Proposition 3 of [19] one can realise that $\text{diag}(\sum_{n=1}^{N} \boldsymbol{w}_n \boldsymbol{w}_n^{\mathcal{H}} \boldsymbol{D}_n)$ is the subgradient of $f(\boldsymbol{Q})$. In particular, the subgradient of $\mu_i = [\boldsymbol{Q}]_{ii}$ at the ith relay is $\sum_{n=1}^{N} |w_{n,i}|^2 [\boldsymbol{D}_n]_{ii}$. Compared to the result in Proposition 3 of [19] the difference here is the inclusion of $\boldsymbol{D}_n$ in the subgradient, a direct consequence of the appearance of $\boldsymbol{D}_n$ in the objective function of (37). □

Having derived the subgradient of $f(\boldsymbol{Q})$, $\boldsymbol{Q}$ is then updated by applying the Euclidean projection $\mathcal{P}_{\mathcal{S}_Q}$ of the subgradient of $f(\boldsymbol{Q})$ on the constraint set $\mathcal{S}_Q = \{\boldsymbol{Q}: \text{tr}(\boldsymbol{QP}) \leq \text{tr}(\boldsymbol{P}), \boldsymbol{Q} \succeq \boldsymbol{0}\}$, that is

$$\boldsymbol{Q}^{(t+1)} = \mathcal{P}_{\mathcal{S}_Q} \left\{ \boldsymbol{Q}^{(t)} + a_t \,\text{diag}\left(\sum_{n=1}^{N} \boldsymbol{w}_n \boldsymbol{w}_n^{\mathcal{H}} \boldsymbol{D}_n \right) \right\} \tag{38}$$

where a_t is an appropriate step size. This subgradient projection method is guaranteed to converge to the global optimum of $f(\boldsymbol{Q})$ [20]. We now summarise the iterative algorithm to solve the distributed beamforming problem with per-relay constraints with the property of distributed implementation as follows:

1. Initialise $\boldsymbol{Q}^{(t)}$. Set $t = 1$.
2. Repeat: fix $\boldsymbol{Q}^{(t)}$, then the relays transmit $\boldsymbol{Q}^{(t)}$ to every destination. Each destination then solves the fixed point iteration in (35) to determine the required power $\lambda_n \sigma_D^2$ for its corresponding virtual uplink channel. The optimal receive beamformer $\hat{\boldsymbol{w}}_n$ and the scaling factor ζ_n are then determined by the nth destination.
3. The nth destination broadcasts λ_n and ζ_n back to relays. The ith relay calculates the beamforming coefficients $w_{1i}, w_{2i}, \ldots, w_{Ni}$ with local information pertaining to the relay as

$$w_{n,i} = \frac{\sqrt{\zeta_n} f_{n,i}^\star g_{i,n}^\star}{\sigma_D^2 [(\sigma_{S_n}^2 |f_{n,i}|^2 + \sigma_R^2)\mu_n + \lambda_n \sigma_R^2 |g_{i,n}|^2]}$$

4. The relays cooperates with each other to update $\boldsymbol{Q}^{(t)}$ as in (38).

IET Commun., 2011, Vol. 5, Iss. 14, pp. 2040–2051
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5. Set $t = t + 1$ and return to Step 2 until convergence.

It is worth nothing that the two proposed algorithms for power minimisation (with and without per-relay power constraints) rely on the simple fixed-point iteration. This iteration is generally very fast in convergence, and also easy to implement in practical systems, instead of utilising an external optimisation software package.

4 Joint SNR-margin-maximised power allocation

4.1 Sum relay power constraint

This section considers the power allocation scheme to jointly maximise the minimal SNR margin subject to a sum relay power constraint. Under such a constraint, the relays are allowed to share a common power pool, although they do not necessarily share their received signals from the sources. The problem is stated as

$$\begin{aligned} &\underset{\boldsymbol{w}_1,\ldots,\boldsymbol{w}_N}{\text{maximise}} \ \min_n \frac{\text{SNR}_n}{\gamma_n} \\ &\text{subject to } P_{\text{relay}} \le P_{\text{relay}}^{\max} \end{aligned} \tag{39}$$

Here, the sum relay power constraint is a firm system restriction. Even so, the problem is always feasible as it is always possible to scale $\boldsymbol{w}_n$ down to meet the sum relay power constraint. The parameter γ_n is interpreted as the weight for user-n's SNR. By introducing the auxiliary variable τ, denoted as the SNR margin, the problem can be reformulated as

$$\begin{aligned} &\underset{\tau,\boldsymbol{w}_1,\ldots,\boldsymbol{w}_N}{\text{maximise}} \ \tau \\ &\text{subject to } \frac{\sigma_{S_n}^2 |\boldsymbol{h}_n^{\mathcal{H}} \boldsymbol{w}_n|^2}{\sigma_R^2 \|\boldsymbol{G}_n^{1/2}\boldsymbol{w}_n\|^2 + \sigma_D^2} \ge \tau\gamma_n, \quad \forall n \\ &\qquad \sum_{n=1}^{N} \boldsymbol{w}_n^{\mathcal{H}} \boldsymbol{D}_n \boldsymbol{w}_n \le P_{\text{relay}}^{\max} \end{aligned} \tag{40}$$

Note that since τ is a variable, the SNR constraint is no longer convex [4]. As a result, problem (40) is not a convex problem. However, for a fixed value of τ, the problem can be formulated as a convex feasibility problem and is readily solved by the bisection method [20]. This is presented next.

4.1.1 Bisection method: Define $\tau^\star$ as the maximum attained value of τ. For a specific target value of τ, the following SOCP feasibility problem is considered

$$\begin{aligned} &\text{find} \quad \boldsymbol{w}_1, \ldots, \boldsymbol{w}_N \\ &\text{subject to } \sqrt{\frac{\sigma_{S_n}^2}{\tau\gamma_n}} \boldsymbol{h}_n^{\mathcal{H}} \boldsymbol{w}_n \ge \left\| \begin{matrix} \sigma_R \boldsymbol{G}_n^{1/2} \boldsymbol{w}_n \\ \sigma_D \end{matrix} \right\|, \quad \forall n \\ &\qquad \sum_{n=1}^{N} \boldsymbol{w}_n^{\mathcal{H}} \boldsymbol{D}_n \boldsymbol{w}_n \le P_{\text{relay}}^{\max} \end{aligned} \tag{41}$$

If the problem is feasible, which means that $\tau < \tau^\star$, then it is possible to increase the target margin τ. Otherwise, if $\tau < \tau^\star$, the target margin should be reduced. The bisection method [20] is summarised as follows:

1. Initialise u and l as the upper and lower bounds of τ.
2. Repeat: $\tau = (u + l)/2$. Solve the feasibility problem (41).
3. If (41) is feasible, then set $l = \tau$, else, set $u = \tau$.
4. Return to Step 2 until $u - l < \varepsilon$,

where ε is a small positive value.

4.1.2 Convex solution: In the previous section, an SOCP approach using bisection method is presented to solve the joint SNR margin maximisation problem. However, such an approach is computationally expensive and unappealing, since it requires many iterations in the bisection method as well as a standard conic solution package. Now, we make use of Lemma 1 to cast problem (40) as a convex optimisation problem with $\tau, p_1, \ldots, p_N$ as the variables

$$\mathcal{S}(P_{\text{relay}}^{\max}) = \begin{cases} \underset{p_1,\ldots,p_N,\tau}{\text{maximise}} & \tau \\ \text{subject to} & \tau\gamma_n - \displaystyle\sum_{i=1}^{R} \frac{a_{n,i}\, p_n}{b_{n,i} + p_n} \le 0, \quad \forall n \\ & \displaystyle\sum_{n=1}^{N} p_n \le P_{\text{relay}}^{\max} \end{cases} \tag{42}$$

The above problem can be solved by any standard convex optimisation algorithm. On the other hand, the connection between the power minimisation with guaranteed QoS problem presented in Section 3.1 and the joint SNR margin maximisation $\mathcal{S}(P_{\text{relay}}^{\max})$ in (42) can be exploited to directly solve (42). This is described next.

4.1.3 Modified fixed point iteration for finding $p_n^\star$: We now establish the connection between power minimisation problem $\mathcal{P}_n(\gamma_n)$ in (16) and the joint SNR margin maximisation problem $\mathcal{S}(P_{\text{relay}}^{\max})$ in the following lemma.

Lemma 4: The joint SNR margin maximisation problem (42) and the power minimisation problem in (16) are inverse problems

$$\tau = \mathcal{S}\left(\sum_{n=1}^{N} \mathcal{P}_n(\tau\gamma_n)\right) \tag{43}$$

$$P_{\text{relay}}^{\max} = \sum_{n=1}^{N} \mathcal{P}_n(\gamma_n \mathcal{S}(P_{\text{relay}}^{\max})) \tag{44}$$

Proof: This lemma is proved by contradiction and by the monotonicity of the function $\Sigma_{i=1}^{R}(a_{n,i}\, p_n / b_{n,i} + p_n)$. Beginning with (43), suppose that $p_n^\star$ is the optimal value and also the optimal argument of $\mathcal{P}_n(\tau\gamma_n)$, then $\Sigma_{i=1}^{R}(a_{n,i}\, p_n^\star / b_{n,i} + p_n^\star) = \tau\gamma_n$. Also, let $\tilde{\tau}^\star$ and $\tilde{p}_n^\star, \forall n$ be the optimal value and arguments of $\mathcal{S}(\Sigma_{n=1}^{N} p_n^\star)$. If $\tilde{\tau}^\star < \tau$, there is a contradiction that $p_n^\star$'s are also feasible solution for $\mathcal{S}(\Sigma_{n=1}^{N} p_n^\star)$, and yet provide a higher objective value τ. On the other hand, if $\tilde{\tau}^\star < \tau$, there is a contradiction that $\tilde{p}_n^\star > p_n^\star$ to make $\tilde{\tau}^\star\gamma_n > \tau\gamma_n$, then the constraint $\Sigma_{n=1}^{N} \tilde{p}_n^\star \le \Sigma_{n=1}^{N} p_n^\star$ in $\mathcal{S}(\Sigma_{n=1}^{N} p_n^\star)$ cannot be true. The proof for (44) follows the same line. □

Using the results in Lemma 4, problem $\mathcal{S}(P_{\text{relay}}^{\max})$ in (42) can be solved by iteratively solving N problems $\mathcal{P}_n(\tau\gamma_n)$ for different values of τ until $\Sigma_{n-1}^{N} \mathcal{P}_n(\tau\gamma_n) = P_{\text{relay}}^{\max}$. Then the optimal arguments $p_n^\star$ of $\mathcal{P}_n(\tau^\star\gamma_n)$ are also optimal to $\mathcal{S}(P_{\text{relay}}^{\max})$. Therefore $p_n^\star$ must also satisfy the fixed point iteration (18) with γ_n replaced by $\tau^\star\gamma_n$. Unfortunately, $\tau^\star$ needs to be determined as well. However, the condition on optimality $\Sigma_{n=1}^N p_n^\star = P_{\text{relay}}^{\max}$ allows a modified fixed point iteration to overcome this difficulty as follows. Let

$$\tilde{p}_n = \frac{\gamma_n}{\Sigma_{i=1}^R (a_{n,i}/b_{n,i} + p_n^{(t)})} \tag{45}$$

Then normalise the result such that the sum relay power is equal to the maximum allowable power

$$p_n^{(t+1)} = \frac{P_{\text{relay}}^{\max}}{\Sigma_{l=1}^N \tilde{p}_l} \tag{46}$$

The iteration gets back to (45) until convergence. The convergence analysis of the proposed algorithm is similar to those in Section 3.1, based on the standard function approach proposed in [18]. Numerous simulations show a rapid convergence rate of the modified fixed point iteration.

4.2 Per-relay power constraints

With the same arguments as in Section 3.2, it might be desirable to have the power constraint at each relay. In this section, with such strict constraints imposed, we examine the optimal power allocation scheme to jointly maximise the joint SNR margin at the destinations. This problem is stated as

$$\begin{aligned} &\underset{\boldsymbol{w}_1,\ldots,\boldsymbol{w}_N}{\text{maximise}} \ \min_n \frac{\text{SNR}_n}{\gamma_n} \\ &\text{subject to } P_i \le P_i^{\max}, \quad \forall i \end{aligned} \tag{47}$$

By introducing the auxiliary variable τ, this problem is restated as

$$\begin{aligned} &\underset{\tau,\boldsymbol{w}_1,\ldots,\boldsymbol{w}_N}{\text{maximise}} \ \tau \\ &\text{subject to } \frac{\sigma_{S_n}^2 |\boldsymbol{h}_n^{\mathcal{H}} \boldsymbol{w}_n|^2}{\sigma_R^2 \|\boldsymbol{G}_n^{1/2}\boldsymbol{w}_n\|^2 + \sigma_D^2} \ge \tau\gamma_n, \quad \forall n \\ &\sum_{n=1}^N \boldsymbol{w}_n^{\mathcal{H}} \boldsymbol{D}_n \boldsymbol{E}_i \boldsymbol{w}_n \le P_i^{\max}, \quad \forall i \end{aligned} \tag{48}$$

4.2.1 Bisection method: Similar to the optimisation problem (40), problem (47) is not convex. However, with a fixed value of τ, the problem can be formulated as a convex feasibility problem. Thus, this problem can be solved by the bisection method [20]. Define $\tau^\star$ as the maximum attained value of τ. For a specific value of τ, the following convex SOCP feasibility problem is considered:

$$\begin{aligned} &\text{find} \quad \boldsymbol{w}_1, \ldots, \boldsymbol{w}_N \\ &\text{subject to } \sqrt{\frac{\sigma_{S_n}^2}{\tau\gamma_n}} \boldsymbol{h}_n^{\mathcal{H}} \boldsymbol{w}_n \ge \left\| \begin{matrix} \sigma_R \boldsymbol{G}_n^{1/2} \boldsymbol{w}_n \\ \sigma_D \end{matrix} \right\|, \quad \forall n \\ &\sum_{n=1}^N \boldsymbol{w}_n^{\mathcal{H}} \boldsymbol{D}_n \boldsymbol{E}_i \boldsymbol{w}_n \le P_i^{\max}, \quad \forall i \end{aligned} \tag{49}$$

If the problem is feasible, then $\tau < \tau^\star$. Otherwise $\tau > \tau^\star$. The bisection method is the same as the one in Section 4.1.1. It should be pointed out that instead of solving the feasibility problem in (49), one can solve the total power minimisation problem under per-relay power constraints (20) with $\tau\gamma_1, \ldots, \tau\gamma_N$ as the target SNR at the destinations:

$$\begin{aligned} &\underset{\alpha,\boldsymbol{w}_1,\ldots,\boldsymbol{w}_N}{\text{minimise}} \ \alpha \sum_{i=1}^R P_i^{\max} \\ &\text{subject to } \sqrt{\frac{\sigma_{S_n}^2}{\tau\gamma_n}} \boldsymbol{h}_n^{\mathcal{H}} \boldsymbol{w}_n \ge \left\| \begin{matrix} \sigma_R \boldsymbol{G}_n^{1/2} \boldsymbol{w}_n \\ \sigma_D \end{matrix} \right\|, \quad \forall n \\ &\sum_{n=1}^N \boldsymbol{w}_n^{\mathcal{H}} \boldsymbol{D}_n \boldsymbol{E}_i \boldsymbol{w}_n \le \alpha P_i^{\max}, \quad \forall i \end{aligned} \tag{50}$$

The resultant optimal $\alpha^\star$ could be used to determine the feasibility of the distributed beamforming problem with the target SNR margin τ. More specifically, if the optimal $\alpha^\star > 1$, it means that at least one of the per-relay power constraints is violated, that is, it is infeasible to meet the SNR targets $\tau\gamma_1, \ldots, \tau\gamma_N$ without compromising the per-relay power constraints. Thus, the target margin τ needs to be adjusted to a smaller value. Conversely, if $\alpha^\star < 1$, one can scale up the beamforming vector $\boldsymbol{w}_n$ to improve the target SNR margin τ without violating the per-relay power constraints. This suggests that at the optimal SNR margin target $\tau^\star$, $\alpha^\star = 1$. Thus, the aforementioned bisection method can be modified to solve problem (50) with different target SNR margin until $\alpha^\star = 1$. The algorithm proposed in Section 3.2 can be readily used to quickly solve (50).

4.2.2 Iterative algorithm for finding $\boldsymbol{w}_n^\star$: In this section, by adapting the algorithm outlined in Section 3.2, a novel iterative algorithm is proposed to directly solve the joint SNR margin maximisation problem (47). It should be emphasised that unlike the iterative algorithm in finding the optimal distributed beamforming design with a fixed target SNR at each destination, the SNR at the nth destination $\tau\gamma_n$ is now a variable. More specifically, τ is an optimisation variable, not a parameter, and has to be determined as well. At first, we revisit the optimisation problem (50). Note that strong duality holds for the optimisation problem (50), that is, $\alpha^\star \Sigma_{i=1}^R P_i^{\max} = \Sigma_{n=1}^N \lambda_n^\star \sigma_D^2$. Furthermore, the bisection method in Section 4.2.1 states that at the optimal value $\tau^\star$, $\alpha^\star = 1$, which also means that $\Sigma_{n=1}^N \lambda_n^\star \sigma_D^2 = \Sigma_{n=1}^N P_i^{\max}$ at optimum. This can be met by adjusting the fixed point iteration in (35). In the algorithm presented here to jointly maximise the SNR margin with per-relay power constraints, the modified fixed point iteration is taken at Step 2. The algorithm is as follows:

1. Initialise $\boldsymbol{Q}^{(t)}$. Set $t = 1$.
2. Repeat: fix $\boldsymbol{Q}^{(t)}$, solve the fixed point λ_n by iterative function

$$\tilde{\lambda}_n = \frac{\gamma_n}{\sigma_D^2 \Sigma_{i=1}^R (a_{n,i}/b_{n,i}\mu_i + \sigma_D^2 \lambda_n)} \tag{51}$$

then normalise the result

$$\lambda_n = \frac{\tilde{\lambda}_n \Sigma_{i=1}^R P_i^{\max}}{\sigma_D^2 \Sigma_{l=1}^N \tilde{\lambda}_l} \tag{52}$$

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so that $\Sigma_{n=1}^{N}\lambda_n\sigma_D^2 = \Sigma_{i=1}^{R}P_i^{\max}$, then return to (51) until convergence.

3. Find the optimal receive beamformers of the virtual channels as

$$\hat{\boldsymbol{w}}_n = (\sigma_D^2\boldsymbol{D}_n\boldsymbol{Q}^{(t)} + \lambda_n\sigma_D^2\sigma_R^2\boldsymbol{G}_n)^{-1}\boldsymbol{h}_n \tag{53}$$

4. Determine the achievable SNR of the virtual uplink channel for each user

$$\tilde{\gamma}_n = \frac{\lambda_n\sigma_{S_n}^2|\hat{\boldsymbol{w}}_n^{\mathcal{H}}\boldsymbol{h}_n|^2}{\hat{\boldsymbol{w}}_n^{\mathcal{H}}\boldsymbol{D}_n\boldsymbol{Q}^{(t)}\hat{\boldsymbol{w}}_n + \lambda_n\sigma_R^2\hat{\boldsymbol{w}}_n^{\mathcal{H}}\boldsymbol{G}_n\hat{\boldsymbol{w}}_n} \tag{54}$$

5. Update the distributed downlink beamformers $\boldsymbol{w}_n = \sqrt{\zeta_n}\hat{\boldsymbol{w}}_n$, where

$$\zeta_n = \frac{\tilde{\gamma}_n\sigma_D^2}{\sigma_{S_n}^2|\boldsymbol{h}_n^{\mathcal{H}}\hat{\boldsymbol{w}}_n|^2 - \tilde{\gamma}_n\sigma_R^2\hat{\boldsymbol{w}}_n^{\mathcal{H}}\boldsymbol{G}_n\hat{\boldsymbol{w}}_n} \tag{55}$$

6. Update $\boldsymbol{Q}^{(t)}$ using subgradient projection method with step size a_t:

$$\boldsymbol{Q}^{(t+1)} = \mathcal{P}_{\mathcal{S}_Q}\left\{\boldsymbol{Q}^{(t)} + a_t\,\mathrm{diag}\left(\sum_{n=1}^{N}\boldsymbol{w}_n\boldsymbol{w}_n^{\mathcal{H}}\boldsymbol{D}_n\right)\right\} \tag{56}$$

7. Set $t \leftarrow t+1$ and return to Step 2 until convergence.

Although the above algorithm takes several steps to find to the optimal solution, the calculation at each step is quite simple and devised specifically for this problem. As a result, the complexity of this approach is fairly low with short running time. In fact, numerous simulations with random channel realisations show a much faster convergence time of the proposed algorithm as compared to the bisection method in Section 4.2.1.

5 Numerical results

This section presents the numerical results on the relay power consumptions to maintain guaranteed QoS at the destinations and on the achievable SNR margin when strict power constraints are imposed at the relays. Also presented are the convergence plots of the proposed iterative algorithms. The network being considered is equipped with four relays. The number of users to be served by the network is 3. The source power is set at 10 for all the users' sources in all the simulations and the noise variances σ_R^2 and σ_D^2 are set to unity. Flat Rayleigh fading is assumed in all the channels, where each $S \to R$ and $R \to D$ channel coefficients is assumed to be i.i.d. $\mathcal{CN}(0, 1)$. When the per-relay power constraints are imposed, the maximum per-relay power is set at 10. The target SNR γ_n is set at 5 for all the destinations.

5.1 Relay power consumptions with guaranteed QoS at destinations

Fig. 2 illustrates the power consumptions at the relays for 50 different channel realisations. At each channel realisation, the sum relay power, and the highest relay power level of the four relays are plotted and compared between the two relaying schemes: with and without per-relay power constraints. As can be seen from the figure, imposing the per-relay power constraints does increase the sum relay power, compared with the optimal scheme that does not impose the constraints. However, the chief advantage of applying the per-relay power constraints is that it balances the power consumption at the relays and does not overuse any of them. Consequently, the highest relay power level of the four relays with the per-relay power constraints is always smaller than that without the constraints.

The convergence of the proposed algorithms is illustrated in Figs. 3 and 4. We utilise the following randomised channel realisation to generate both figures (see (57))

Fig. 3 plots the evolution of the sum relay power allocated for user-1, p_1 and the corresponding SNR_1 after each iteration

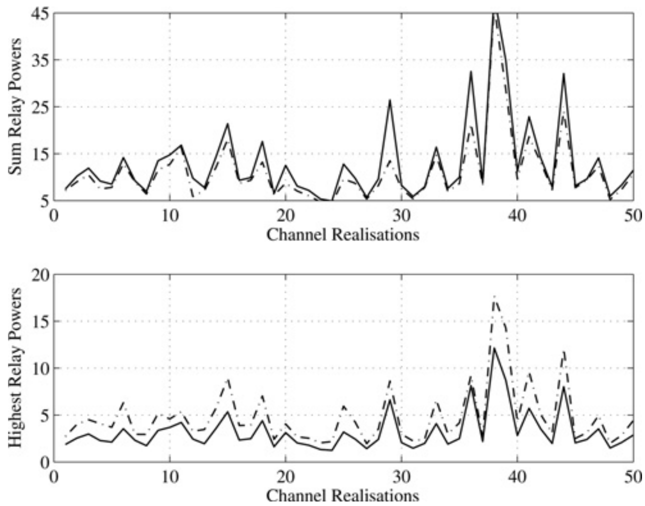

Fig. 2 *Power consumptions at the relays over 50 channel realisations with different power constraints: with per-relay power constraints (solid lines), without per-relay power constraints ('dash–dot' lines)*

$$\begin{aligned}
[\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3] &= \begin{bmatrix} 1.384 + 0.060\mathrm{i} & -0.805 - 0.227\mathrm{i} & 0.449 - 0.870\mathrm{i} \\ 0.356 - 1.417\mathrm{i} & -0.149 + 0.874\mathrm{i} & -0.425 + 0.746\mathrm{i} \\ 1.318 - 0.348\mathrm{i} & 0.841 - 0.446\mathrm{i} & 0.389 - 0.080\mathrm{i} \\ -0.240 + 0.326\mathrm{i} & -0.789 - 1.644\mathrm{i} & -0.777 + 0.268\mathrm{i} \end{bmatrix} \\
[\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3] &= \begin{bmatrix} 0.667 + 0.415\mathrm{i} & -0.719 - 0.055\mathrm{i} & 0.867 - 0.008\mathrm{i} \\ -1.499 - 0.177\mathrm{i} & -0.128 + 0.628\mathrm{i} & -0.492 + 0.645\mathrm{i} \\ -0.455 + 0.339\mathrm{i} & 1.075 + 1.632\mathrm{i} & 0.005 + 0.039\mathrm{i} \\ -0.498 + 0.472\mathrm{i} & -0.027 + 0.371\mathrm{i} & -0.553 - 0.782\mathrm{i} \end{bmatrix}
\end{aligned} \tag{57}$$

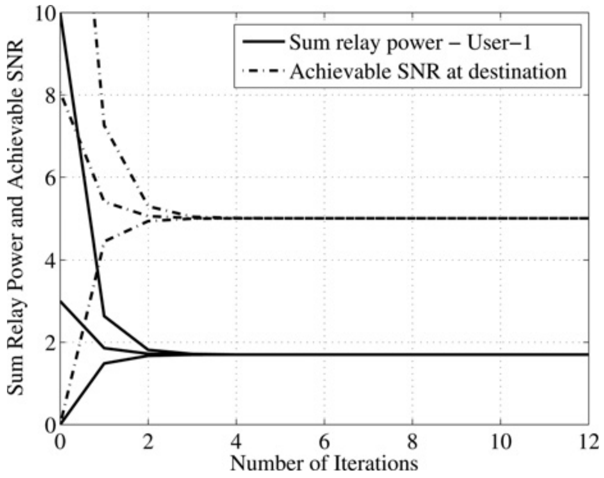

Fig. 3 *Convergence of the iterative fixed point algorithm (18) with different starting points and the achievable SNR at user-1's destination after each iteration*

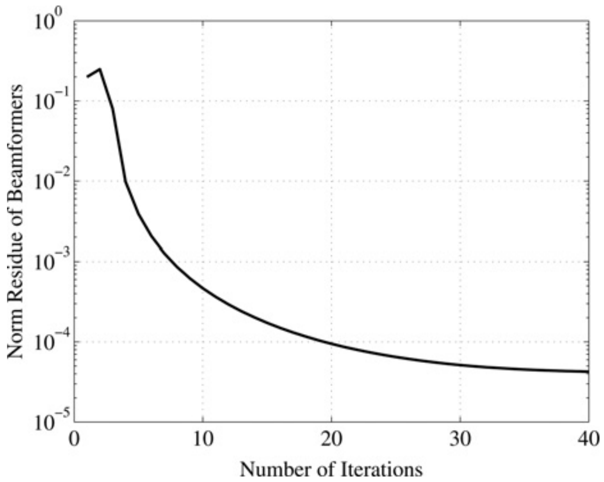

Fig. 4 *Convergence of the proposed algorithm in finding the optimal distributed beamformers with per-relay power constraints*

by the iterative fixed point algorithm (18). It can be seen that the algorithm converges very quickly after only a few iterations to the optimal $p_1^\star$ from various arbitrary starting points, whereas the corresponding SNR also converges to its target value $\gamma_1 = 5$. Fig. 4 displays the convergence of the proposed iterative algorithm in Section 3.2.3 in finding the optimal distributed beamformers $\boldsymbol{w}_n^\star$ with per-relay power constraints. The step-size $a_t = 1/t$ is used for the subgradient update of the iterative algorithm. The summation $\sum_{n=1}^{N} \|\boldsymbol{w}_n - \boldsymbol{w}_n^\star\|$, which is the norm residue of the beamformers, plotted after each iteration clearly shows the convergence of the proposed algorithm. Numerous simulations also show that the proposed algorithm converges in a small fraction of the running time required by the cvx package [14].

5.2 Achievable SNR margin

Fig. 5 illustrates the achievable SNR margin at the destination, that is, $\min_n \mathrm{SNR}_n/\gamma_n$ for 50 different channel realisations. At each channel realisation, lowest SNR margin, the highest relay power levels of the four relays are plotted and compared between two different constraints: sum relay power constraint and per-relay power constraints. As can be observed in the figure, imposing the per-relay

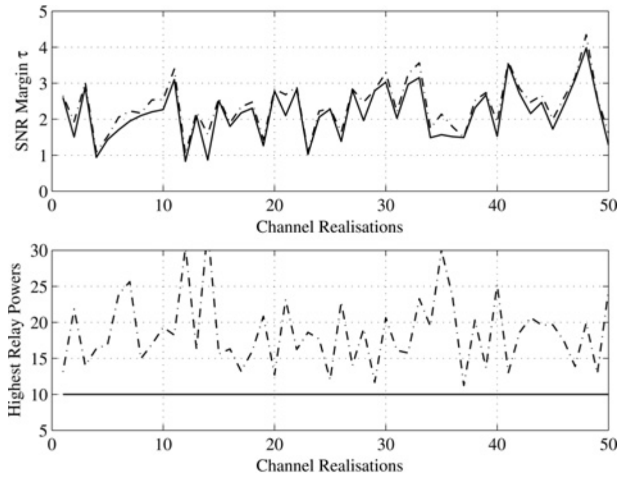

Fig. 5 *Achievable SNR margin τ and the power consumptions at the relays over 50 channel realisations with different power constraints: with per-relay power constraints (solid lines), and without per-relay power constraints ('dash–dot' lines)*

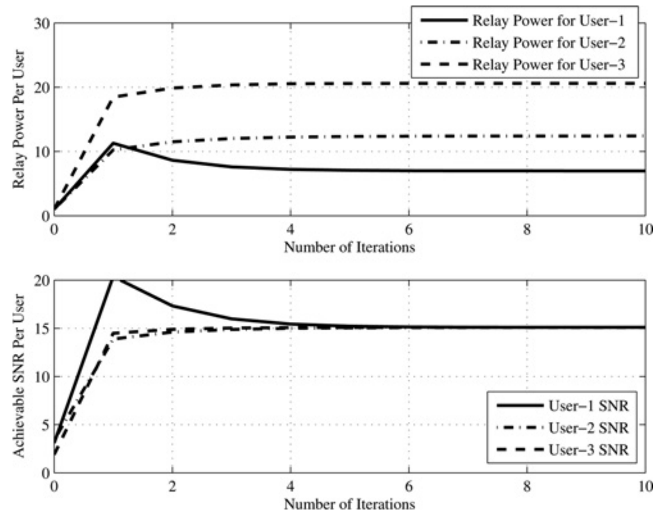

Fig. 6 *Convergence of the relay power for each user and the corresponding achievable SNR at each user's destination by the modified iterative fixed point algorithm*

power constraints does not decrease achievable SNR margin much in most of the simulations, compared with the optimal scheme that imposes the sum relay power constraint. To loosely explain this observation, it is noticed that the objective function (the minimum SNR margin) is rather flat near its optimality, such that a change in the power allocation at each relay does not change the objective function much. However, at the optimal solution, the allocated power may be highly inclined to one of the relays, which can be seen from the figure. Although the highest transmit power at each relay is strictly under or equal to 10 with the per-relay power constraints, it may reach to 25 with the sum power constraint. This is a clear benefit of imposing the per-relay power constraints in terms of power consumption at each relay.

Fig. 6 illustrates the convergence of the modified fixed point algorithm in Section 4.1.3. Plotted are the evolution of the allocated relay power p_n and the corresponding SNR_n, $n = 1, 2, 3$ for the three users after each iteration. It can be seen that the algorithm converges very quickly after only a few iterations, as the allocated relay power for each

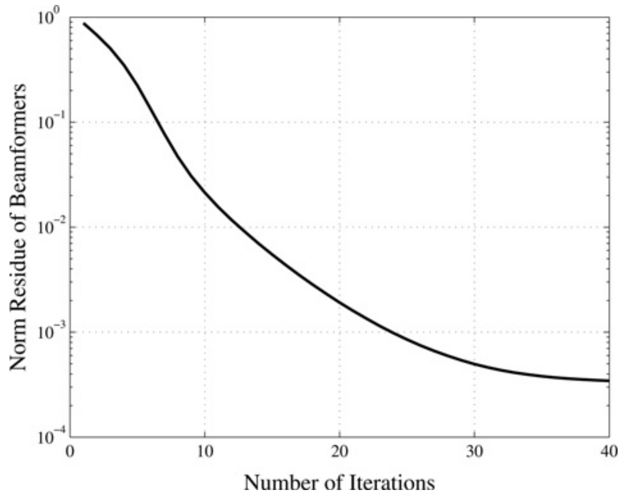

Fig. 7 *Convergence of the proposed algorithm in finding the optimal distributed beamformers with per-relay power constraints to jointly maximise the SNR margin*

user converges to its optimal value. The corresponding SNR also converges to the same optimal value, as all users' SNRs are set at the same weight.

Finally, Fig. 7 displays the convergence of the proposed iterative algorithm in Section 4.2.2 in finding the optimal distributed beamformers $\boldsymbol{w}_n^\star$ with per-relay power constraints to maximise the SNR margin. Again, the summation $\sum_{n=1}^{N} \|\boldsymbol{w}_n - \boldsymbol{w}_n^\star\|$ plotted after each iteration clearly shows the convergence of the proposed algorithm. It is noted that the same randomised channel realisation in (57) is also applied to obtain Figs. 6 and 7.

6 Conclusions

In this paper, we have studied the optimal power allocation schemes in a multiuser multi-relay network to either minimise the total relay power with guaranteed QoS at the destinations or maximise the SNR margin subject to power constraints at the relays. By means of convex optimisation techniques, it was shown that these problems can be formulated and solved via SOCP. Optimal solutions to the two problems can be obtained by any conic solution package. In addition, by applying the fixed-point iteration framework to the relay network, we also proposed simple and fast iterative algorithms to directly solve the two optimisation problems. As the proposed algorithms work without the need of external software package, they can be easily implemented and are more suitable in real-time communications.

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