

# Orthogonal rational approximation of transfer functions for high-frequency circuits

Andrew Ma | Arif Ege Engin 

Department of Electrical and Computer Engineering, San Diego State University, San Diego, California, USA

## Correspondence

Arif Ege Engin, Department of Electrical and Computer Engineering, San Diego State University, San Diego, CA, USA.  
Email: [aengin@sdsu.edu](mailto:aengin@sdsu.edu)

## Summary

This paper introduces the orthogonal rational approximation (ORA) algorithm for rational function approximation of transfer functions, based on data available from simulations or measurements. A rational function allows integration of such data in transient solvers for analysis of high-frequency circuits. In rational function approximation, the unknown denominator polynomial of the model results in a nonlinear problem, which can be replaced with successive solutions of linearized problems following the Sanathanan–Koerner (SK) iteration. An orthogonal basis can be obtained based on Arnoldi resulting in a stabilized SK iteration. We present an extension of the stabilized SK, called ORA, which ensures real polynomial coefficients and stable poles for realizability of electrical networks. We also introduce an efficient implementation of ORA for multiport networks based on a block QR decomposition.

## KEYWORDS

macromodeling, rational function, transfer function, vector fitting

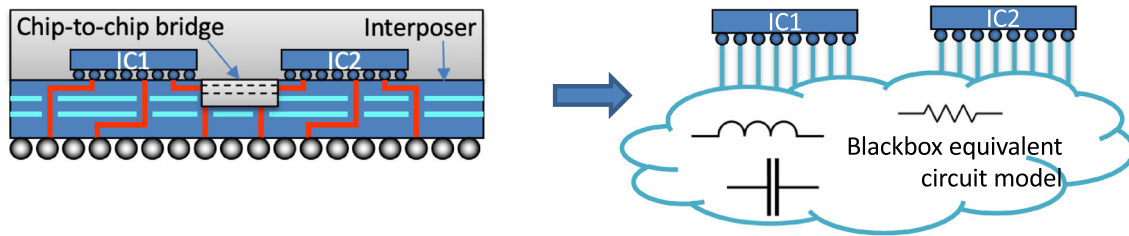
## 1 | INTRODUCTION

Rational function approximations find applications in many areas including macromodeling of high-frequency circuits,<sup>1</sup> model order reduction for controller design,<sup>2</sup> interpolation and extrapolation of system responses,<sup>3</sup> surrogate models for high-energy physics,<sup>4</sup> and approximation of elementary mathematical functions.<sup>5</sup>

Electromagnetic modeling of microelectronics packaging is a large field that benefits from accurate rational function approximations. Slow-down of Moore's law and economical concerns of yield are pushing the semiconductor industry toward heterogeneous integration, where multiple dies are interconnected through an advanced chip package, resulting in a system-in-package (SiP). Heterogeneous integration allows a “more-than-Moore” approach that enables cutting-edge computing.<sup>6–10</sup> In such advanced packaging, the interconnect parasitics can no longer be modeled using isolated circuit models available in closed-form<sup>11,12</sup> requiring blackbox models for time-domain circuit simulation of complex electromagnetic systems as shown in Figure 1. Their description is however generally available as scattering parameters obtained from simulations or measurements. An intermediate step in generating an equivalent circuit model is a rational function approximation of this tabulated data.<sup>13–16</sup> Available methods for rational transfer function approximation include the widely popular vector fitting,<sup>1</sup> Loewner framework,<sup>17</sup> Sanathanan–Koerner (SK) iteration,<sup>18</sup> and AAA (adaptive Antoulas–Anderson).<sup>19–22</sup>

This is an open access article under the terms of the [Creative Commons Attribution-NonCommercial](https://creativecommons.org/licenses/by-nc/4.0/) License, which permits use, distribution and reproduction in any medium, provided the original work is properly cited and is not used for commercial purposes.

© 2022 The Authors. *International Journal of Circuit Theory and Applications* published by John Wiley & Sons Ltd.



**FIGURE 1** Time-domain circuit simulation for signal and power integrity design of a system in package (SiP) requires a blackbox equivalent circuit model [Colour figure can be viewed at [wileyonlinelibrary.com](http://wileyonlinelibrary.com)]

Vector fitting is related to SK iteration.<sup>23</sup> The partial fractions basis used in vector fitting and the iterative pole relocation has resulted in a robust method with wide range of successful applications. The implementation of SK iteration based on a monomial basis, on the other hand, becomes severely ill-conditioned due to two major reasons: the presence of Vandermonde matrices, and the weighting introduced by multiplication with the denominator from the previous iteration. One remedy to improve the conditioning of the method is to use orthogonal polynomials such as Chebyshev polynomials,<sup>24</sup> referred to as generalized SK iteration,<sup>25</sup> and to properly scale the frequency variable. An orthogonal basis can also be generated on-the-fly at arbitrary frequency points using Vandermonde with Arnoldi.<sup>26</sup> Similar approaches with orthogonal polynomial bases have also been studied for rational function approximation.<sup>4,27–30</sup> A solution to address the second major source of ill-conditioning due to the weighting is introduced in the stabilized SK iteration.<sup>5</sup>

A vector fitting method based on an orthogonal basis of partial fractions is also available.<sup>31,32</sup> Our method is not based on a partial fractions basis; therefore, it is numerically different, and we argue that it is simpler, especially in enforcing real coefficients for the polynomials. An advantage of our method is its flexibility to start the SK iteration with an arbitrary denominator polynomial in addition to the usual selection of initial poles in vector fitting. This would for example allow to start the iteration with an initial choice of a denominator polynomial for numerical stability.<sup>33</sup> It is also possible to fit rational functions with a relative degree greater than 1.

Using an orthogonal polynomial basis has been studied for rational function approximation before,<sup>4,27–30</sup> where the coefficients of the orthogonal polynomial basis can be obtained using a three-term recurrence relation. We emphasize that an orthogonal basis is needed for the rational functions and not merely the polynomials. The improvement in accuracy using orthogonal rational functions rather than orthogonal polynomials can be dramatic as we will demonstrate in the numerical results.

Orthogonal rational bases have been used in system identification before, mostly for discrete-time systems,<sup>32,34,35</sup> including approximations with real coefficients.<sup>36</sup> The rational Arnoldi decomposition by Ruhe<sup>37</sup> has also been successfully used to solve more general problems such as approximation of matrix functions or eigenvalues.<sup>38</sup> In particular, the RKFIT algorithm<sup>39</sup> is also based on an orthogonal rational basis to solve the rational approximation problem. Our method generates the orthogonal rational basis using a regular Arnoldi (or Lanczos) algorithm (unlike RKFIT that uses the rational Arnoldi decomposition algorithm), and it includes an efficient multiport implementation. Therefore, we expect a similar accuracy between ORA and RKFIT, whereas our method is expected to be faster.

In prior work, we have compared both polynomial and rational approximation methods for interpolation and least squares problems of scalar functions with no consideration on stability.<sup>40</sup> In this paper, we address the approximation problem of multiport networks with rational functions having real polynomial coefficients and stable poles. We extend the Arnoldi iteration method for calculating an orthogonal basis for the Vandermonde matrix<sup>26</sup> to ensure real coefficients and integrate it in the stabilized SK iteration.<sup>5</sup> We extend this formulation to ensure stable poles and introduce an efficient methodology for multiport networks. The resulting orthogonal rational approximation (ORA) method is suitable for approximation of transfer functions with stable poles from tabulated data of multiport network parameters.

## 2 | RATIONAL APPROXIMATION OF TRANSFER FUNCTIONS

When performing a network parameter approximation using a rational function, the goal is a transfer function expressed as a ratio of two polynomials

$$\frac{n(s)}{d(s)} = \frac{\sum_{i=0}^n a_i s^i}{\sum_{i=0}^d b_i s^i}. \quad (1)$$

Assume that  $m$  data samples  $f_k$  are provided at frequency points  $s_k = j\omega_k$ . A least squares solution is desired that minimizes the residuals  $n(s_k)/d(s_k) - f_k$  at the provided frequency points. This problem is related to model order reduction (MOR) techniques<sup>15,41</sup> such as asymptotic waveform evaluation (AWE) or passive reduced-order interconnect macromodeling algorithm (PRIMA). For rational approximation, we start with measured or simulated data only, whereas MOR requires that there is an existing model with large order. It is possible to sample an existing model and use rational approximation of the sampled data as an MOR method; however, in the following, we assume that only the data are available.

The rational function can be normalized by fixing one of its coefficients such as  $b_0 = 1$ . Another option is fixing the norm of the unknown coefficients resulting in the least-squares problem addressed in this paper

$$\underset{a,b}{\text{minimize}} \sum_{k=1}^m \left| \frac{n(s_k)}{d(s_k)} - f_k \right|^2, \text{ s.t. } \left\| \begin{bmatrix} a \\ b \end{bmatrix} \right\| = 1, \quad (2)$$

which is typically an overdetermined problem, where there are more frequency points than the number of coefficients. This is however a nonlinear problem for the general case of unknown coefficients of the denominator polynomial. The monomial basis in (2) also results in an ill-conditioned problem and even switching to an orthogonal polynomial basis does not result in accurate approximations for large size problems. To overcome the numerical conditioning problem, a stabilized SK iteration<sup>5</sup> has been proposed, which however does not guarantee real coefficients or stable poles. In this paper, we introduce the ORA method that satisfies these properties, which are needed for rational function approximation of realizable networks. ORA can also efficiently find rational approximations of multi-port networks (where the coefficients  $a_i$  are matrices).

### 3 | FITTING THE NUMERATOR POLYNOMIAL

To introduce ORA, we first discuss polynomial approximation, followed by rational approximation with known denominator polynomial.

#### 3.1 | Orthogonal polynomial basis with real coefficients

If  $d(s) = 1$ , the problem reduces to a polynomial approximation. In this case, the Vandermonde matrix  $A$  can be used to solve the coefficients  $a$  of  $n(s)$  that approximates the data in the least-squares sense

$$Aa \approx f. \quad (3)$$

The Vandermonde matrix is of size  $m \times (n+1)$  and given as  $A = [1s^2 \dots s^n]$ , where  $s = (s_1, s_2, \dots, s_m)^T$  is the vector of frequency points. An orthogonal basis for  $A$  can be found with the  $A = QR$  decomposition, which however results in an ill-conditioned least squares problem in (3) for high values of  $n$ .

The Arnoldi iteration provides the same matrix  $Q$  with orthogonal columns  $[q_0 q_1 q_2 \dots q_n]$ , with the advantage that the ill-conditioned matrix  $A$  (and  $R$ ) is never actually formed.<sup>26</sup> It can be observed that the matrix  $A$  is equivalent to

$$A = [q_0 S q_0 S^2 q_0 \dots S^n q_0], \quad (4)$$

where the starting polynomial is  $q_0 = (1, 1, \dots, 1)^T$  and  $S = \text{diag}(s)$ . The Arnoldi iteration is based on the application of Gram-Schmidt (GS) orthogonalization on the columns of  $A$  in sequence. At step  $i$ , the vector  $Sq_{i-1}$  is orthogonalized against the previous columns  $[q_0 q_1 q_2 \dots q_{i-1}]$ . Each column is chosen to have a norm of  $\sqrt{m}$ , to be consistent with the

norm of  $q_0$ . The Arnoldi iteration also provides an  $(n+1) \times n$  upper Hessenberg matrix  $H$  that includes the coefficients used in orthogonalization process such that

$$SQ_- = QH, \quad (5)$$

where  $Q_-$  is obtained by removing the last column of  $Q$ . The polynomial coefficients  $a$  are never calculated; instead, the least-squares problem is solved using transformed coefficients  $c$  as

$$Qc \approx f. \quad (6)$$

Generation of orthogonal polynomials for data fitting has been known for many decades (see, e.g., Forsythe<sup>42</sup>). More recently, the evaluation of such a fitted polynomial (at arbitrary frequency points) has been streamlined<sup>26</sup> without explicitly using the three-term recurrence coefficients. The key insight is the use of  $H$  to that purpose. For evaluating the polynomial at another set of frequencies  $\hat{S}$ , the same operations based on the previously obtained  $H$  are applied to simply obtain the matrix  $\hat{Q}$

$$\hat{S}\hat{Q}_- = \hat{Q}H. \quad (7)$$

The data at this new set of frequencies can then be obtained as  $\hat{Q}c$ .

For realizability of electrical networks, transfer functions with real polynomial coefficients are needed. To ensure real coefficients, one option is to enforce conjugate symmetry by fitting the complex conjugate responses on both sides of the frequency axis,<sup>27,28</sup> which however unnecessarily uses a complex-valued Arnoldi iteration. We present an alternative method that uses real arithmetic throughout to ensure real coefficients by using an updated initial vector and frequency matrix. For the polynomial fitting example, the vector of coefficients  $a$  in (3) can be enforced to be real by stacking the real and imaginary parts of  $A = A' + jA''$  and  $f = f' + jf''$  as

$$\begin{bmatrix} A' \\ A'' \end{bmatrix} a \approx \begin{bmatrix} f' \\ f'' \end{bmatrix}. \quad (8)$$

We can now observe that the stacked matrix is equivalent to

$$\begin{bmatrix} A' \\ A'' \end{bmatrix} = [qXqX^2q\dots X^nq], \quad (9)$$

where

$$q = \begin{bmatrix} q'_0 \\ q''_0 \end{bmatrix}, X = \begin{bmatrix} S' & -S'' \\ S'' & S' \end{bmatrix}, \quad (10)$$

and the real and imaginary parts of  $S$  are given as  $S = S' + jS''$ . The obtained orthogonal basis would also be in stacked form as

$$\begin{bmatrix} Q' \\ Q'' \end{bmatrix}, \quad (11)$$

where  $Q = Q' + jQ''$ . For the usual case of a pure imaginary  $s$  vector,  $X$  becomes skew symmetric (it has zeros on its diagonal), and a Lanczos procedure equivalent to Arnoldi can be obtained.<sup>43</sup> We use the Lanczos procedure for skew-symmetric matrices in this paper. Of particular interest are the zeroes of the orthogonal polynomial, which can be obtained in general from the comrade matrix<sup>44</sup> or the state-space approach<sup>29</sup> as the eigenvalues of the matrix

$$H_n^T - hec_{0:n-1}^T/c_n, \quad (12)$$

where  $H_n$  is an  $n \times n$  matrix obtained from  $H$  by removing its last row,  $h$  is the bottom right element of  $H$ ,  $e$  is a vector of  $n - 1$  zeros followed by a 1 as its last element, and  $c_{0:n-1}$  is the coefficient vector except for the last element  $c_n$ . Since this matrix is real, the obtained zeros are either real or come in complex conjugate pairs.

### 3.2 | Known denominator polynomial

As the next special case, assume that there is an arbitrary denominator polynomial  $d(s)$ , but its value is known at the  $m$  frequency points given by the vector  $d = 1./[d(s_1), d(s_2), \dots, d(s_m)]^T$ , where the division is element-wise. Assume also that  $d$  is normalized to have a norm of  $\sqrt{m}$ . For rational approximation, this case of known denominator would correspond to obtaining the numerator polynomial after the poles have been extracted. This results in the least squares problem of

$$DAa \approx f, \quad (13)$$

where  $D$  is an  $m \times m$  diagonal matrix given as  $D = \text{diag}(d)$ . Using a polynomial orthogonal basis  $Q_p$  as discussed yields

$$DQ_p \hat{c} \approx f, \quad (14)$$

which however can still become ill-conditioned due to the multiplication with  $D$ . The stabilization of SK iteration can be obtained by addressing this problem through generating an orthogonal basis for  $DA$  instead.<sup>5</sup> For a rational function with real coefficients, this can simply be achieved by changing the initial vector in (10) to

$$q = \begin{bmatrix} d' \\ d'' \end{bmatrix} \quad (15)$$

to calculate an orthogonal basis  $Q = Q' + jQ''$  for the rational function and finding the least squares solution as usual from

$$\begin{bmatrix} Q' \\ Q'' \end{bmatrix} c \approx \begin{bmatrix} f' \\ f'' \end{bmatrix}. \quad (16)$$

At this point, if it is desired to calculate the numerator polynomial only, the straightforward option is to use  $D^{-1}Qc$ . However, this is only applicable at the original frequency points. A more powerful alternative is to use the  $H$  matrix to obtain  $\hat{Q}_p$  at arbitrary set of frequency points and calculating  $\hat{Q}_p c$ . Note that  $\hat{Q}_p$  may not be orthogonal even at the original frequency points.

In this section, we have introduced how selecting the initial vector of the Arnoldi iteration from a known denominator polynomial allows us to obtain orthogonal rational functions with real coefficients. The algorithm is implemented in the Matlab function `numfit` as shown in Figure 2. Next, we integrate these results in the SK iteration to obtain the denominator polynomial as well.

## 4 | FITTING THE DENOMINATOR POLYNOMIAL

A simple linearized version of the rational approximation problem<sup>33,45</sup> can be formulated as

$$\underset{a,b}{\text{minimize}} \sum_{k=1}^m |n(s_k) - f_k d(s_k)|^2, \text{ s.t. } \left\| \begin{bmatrix} a \\ b \end{bmatrix} \right\| = 1, \quad (17)$$

whose solution can be obtained from a singular value decomposition (SVD). This naive method may not provide the correct solution due to the linearization. A well-known method to compensate for this inaccuracy is the SK iteration.

```

function [H,Q,fit,myss,err] = numfit(den,s,n,f)
% den: denominator, s=jw, n: num degree, f: data
m = length(s);
Q = [real(den);imag(den)]*sqrt(m)/norm(den);
H = zeros(n+1,n);
for k = 1:n %Lanczos for skew-symmetric matrix
    q1 = Q(1:m,k);
    q2 = Q(m+1:2*m,k);
    q = [-imag(s).*q2; imag(s).*q1];
    if k>1
        H(k-1,k) = -H(k,k-1);
        q = q - H(k-1,k)*Q(:,k-1);
    end
    H(k+1,k) = norm(q)/sqrt(m);
    Q = [Q q/H(k+1,k)];
end
if nargin == 4 %data is provided
    g = Q\[real(f); imag(f)]; %coeffs of numerator
    fit = Q*g;
    fit = fit(1:end/2,:)+ 1i*fit(end/2+1:end,:);
    err = norm(fit(:)-f(:));
    c = Q\[ones(m,1); zeros(m,1)]; %coeffs of denominator
    e=zeros(n,1);
    e(n)=1;
    g=g';
    A=H(1:n,1:n)'-H(n+1,n)*e*c(1:n)'/c(n+1);
    B=H(n+1,n)*e/c(n+1);
    C=g(:,1:end-1)-g(:,end)*c(1:n)'/c(n+1);
    D=g(:,end)/c(n+1);
    myss=ss(A,B,C,D); %state-space assuming improper rational function
end

```

FIGURE 2 Matlab implementation of (16) that fits the numerator with real coefficients from data on the imaginary axis and prescribed denominator [Colour figure can be viewed at [wileyonlinelibrary.com](http://wileyonlinelibrary.com)]

The SK iteration can be started with the linearized least squares problem of (17) to obtain an initial solution for the denominator polynomial  $d_{pre}(s)$ . In order to approach the correct norm, the following problem is then solved to obtain updated  $n(s)$  and  $d(s)$  polynomials:

$$\underset{a,b}{\text{minimize}} \sum_{k=1}^m \left| \frac{n(s_k)}{d_{pre}(s_k)} - f_k \frac{d(s_k)}{d_{pre}(s_k)} \right|^2, \quad (18)$$

with a suitable nontriviality constraint that we will discuss at the end of this section. This process can be continued iteratively until  $d_{pre}(s) \approx d(s)$  so the correct norm is recovered if the algorithm converges.

The ORA is based on generating the orthogonal rational basis  $Q_n$  for  $n(s)/d_{pre}(s)$  and  $Q_d$  for  $d(s)/d_{pre}(s)$ . If the degree of the numerator and denominator polynomials are equivalent,  $Q_n = Q_d$ ; otherwise one can be obtained from the other by removing its last columns depending on the difference in degree. Assuming the diagonal matrix  $F$  contains the data values as  $F = \text{diag}(f_1, f_2, \dots, f_m)$ , the least squares problem can be expressed as

$$\begin{bmatrix} Q'_n & -(FQ_d)' \\ Q''_n & -(FQ_d)'' \end{bmatrix} \begin{bmatrix} g \\ c \end{bmatrix} \approx 0. \quad (19)$$

At each step of the iteration, the roots of  $d_{pre}$  are calculated and any unstable poles are flipped to obtain a set of stable poles  $p_i$  similar to the process in vector fitting. The stable denominator polynomial is then calculated from  $d_{pre}(s_k) = \prod_{i=1}^d w(s_k - p_i)$ , using a roughly chosen weight  $w$  to prevent overflow.

As for preventing the nontrivial solution, the straightforward choice is enforcing that the solution vector in (19) has a norm of 1. The solution is then obtained through the SVD of the matrix in (19).

An alternative is enforcing that condition on only the denominator coefficients  $c$  and not having any constraints on  $g$ . In that case, the denominator polynomial with the coefficients vector  $c$  can be calculated first, followed by calculating the numerator coefficients  $g$  in a second step. Assume we obtain a QR decomposition of the matrix in (19), where the submatrices of  $R$  are given as  $R_{11}, R_{12}, R_{22}$ . The least squares problem with this alternative constraint can then be expressed as

$$\begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix} \begin{bmatrix} g \\ c \end{bmatrix} \approx 0, \text{s.t. } \|c\| = 1. \quad (20)$$

The solution for this homogeneous equation is given by the eigenvector corresponding to the smallest eigenvalue of  $R_{22}^T R_{22}$ .<sup>46</sup> This is equivalent to calculating the final right singular vector of  $R_{22}$  using SVD.

#### 4.1 | Multiport networks

For a multiport network, a common-pole model can be obtained. Assume that  $N$  elements of a network matrix will be approximated using the data set  $F_1, F_2, \dots, F_N$ . Since all  $N$  rational functions will be using the same denominator polynomial, we can fit them simultaneously as

$$\begin{bmatrix} Q'_n & 0 & \dots & 0 & -(F_1 Q_d)' \\ Q''_n & 0 & \dots & 0 & -(F_1 Q_d)'' \\ 0 & Q'_n & \dots & 0 & -(F_2 Q_d)' \\ 0 & Q''_n & \dots & 0 & -(F_2 Q_d)'' \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & Q'_n & -(F_N Q_d)' \\ 0 & 0 & \dots & Q''_n & -(F_N Q_d)'' \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_N \\ c \end{bmatrix} \approx 0. \quad (21)$$

To obtain a fast method for multiport networks, we can enforce  $\|c\| = 1$  as the nontriviality constraint and obtain the denominator first. This method is similar to the handling of multiport networks in the fast implementation of the vector fitting algorithm<sup>47</sup> and the parametric macromodeling approach.<sup>48</sup> The least squares problem then reduces to

$$\begin{bmatrix} R_{22}^1 \\ R_{22}^2 \\ \vdots \\ R_{22}^N \end{bmatrix} c \approx 0, \text{s.t. } \|c\| = 1. \quad (22)$$

We improve the efficiency further by using a block QR decomposition for calculating the  $R_{22}$  terms.<sup>49</sup> Consider the matrix in (19):

$$[Q_1 B] = \begin{bmatrix} Q'_n & -(F Q_d)' \\ Q''_n & -(F Q_d)'' \end{bmatrix}. \quad (23)$$

Since  $Q_1$  is already orthogonal, the thin QR decomposition would be in the form of

$$[Q_1 B] = [Q_1 Q_2] \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix}. \quad (24)$$

Multiplying both sides from the left with  $Q_1^T$  yields  $Q_1^T B = m R_{12}$ , where the factor of  $m$  comes from our choice of having orthogonal columns with a norm of  $\sqrt{m}$ . We can now calculate  $R_{22}$  from the QR decomposition of a smaller matrix  $B - Q_1 Q_1^T B / m = Q_2 R_{22}$ . This is an additional advantage for ORA. The orthogonal rational functions not only improve the numerical conditioning but also speed up the computation for multi-port networks.

The implementation of denominator fitting in ORA is shown in Figure 3.

```

function [den,poles] = denfit(den,s,n,d,f)
%den: denominator, s=jw, n,d: degree of num and den, f: data
m = length(s);
[H,Q] = numfit(den,s,max(n,d));
Qd1 = Q(1:m,1:d+1);
Qd2 = Q(m+1:2*m,1:d+1);
Q = Q(:,1:n+1);
A = [];
for k=1:size(f,2)
    fr=real(f(:,k));
    fi=imag(f(:,k));
    Ar = -[fr.*Qd1-fi.*Qd2;fr.*Qd2+fi.*Qd1];
    Ar = Ar - Q*(Q.'*Ar)/m;
    [~,R] = qr(Ar,0);
    A = [A;R];
end
[~,~,V] = svd(A,0);
c = V(:,end);
poles = eig(H(1:d,1:d) - H(d+1,d)*(1/c(end))*c(1:end-1)*flip(eye(1,d)));
poles(find(real(poles)>0)) = poles(find(real(poles)>0))*-1; %pole flipping
den = 1./((prod((s.'-poles)./mean(abs(s)))));

```

FIGURE 3 Matlab implementation of (22) that updates the denominator [Colour figure can be viewed at [wileyonlinelibrary.com](https://onlinelibrary.wiley.com/doi/10.1002/ca.3488)]

```

function [poles,fit,myss,err] = ora(file,ind,n,d,iter)
% file: touchstone filename, ind: index of matrix elements
% n,d: degree of num and den, iter: iteration count
% example: ora('filename.s2p',[1 1;1 2;2 1;2 2],50,50,20);
S = sparameters(file);
f = [];
for k = 1:size(ind,1)
    f = [f squeeze(S.Parameters(ind(k,1),ind(k,2),:))];
end
frequency = S.Frequencies;
denk = ones(length(frequency),1); %initial denominator is a 1 vector
s=1i*2*pi*frequency;
err = inf;
for k = 1:iter %SK iteration
    [denk,polesk] = denfit(denk,s,n,d,f);
    [~,~,fitk,myssk,errk] = numfit(denk,s,n,f);
    if errk<err %keep the best fit in iterations
        poles=polesk;
        fit=fitk;
        myss=myssk;
        err=errk;
    end
end
if n~=d %state-space implemented only for improper rational functions
    myss=[];
end

```

FIGURE 4 Matlab implementation of ORA [Colour figure can be viewed at [wileyonlinelibrary.com](https://onlinelibrary.wiley.com/doi/10.1002/ca.3488)]

## 5 | ORA

The SK iteration using the `numfit` and `denfit` functions is implemented in the `ora` function as shown in Figure 4. Initially `denfit` is called with a 1 vector for the denominator. At each step, `denfit` is called with the denominator from the previous iteration. We note that the SK iteration in general is not guaranteed to find the optimum solution. If necessary, additional nonlinear optimization algorithms can be applied to find the local minimum starting with the solution obtained from the SK iteration.<sup>5</sup>



## 5.1 | Integration in circuit solvers

If needed, generating a transfer function in pole-residue form is straightforward. Once the denominator polynomial is obtained, the poles of the rational function are readily available from the eigenvalues of (12). The residues can be fit in a similar way to the vector fitting residue identification process in a robust way. Alternatively, a direct state-space representation can be obtained from the coefficients of orthogonal polynomials for the case ( $m < n$ ).<sup>29</sup> This can be extended for the improper rational function ( $m = n$ ) case as

$$\begin{aligned} A &= H_n^T - h e c_{0:n-1}^T / c_n \\ B &= h e / c_n \\ C &= (G_0 G_1 \dots G_{n-1}) - G_n c_{0:n-1}^T / c_n \\ D &= G_n / c_n, \end{aligned} \quad (25)$$

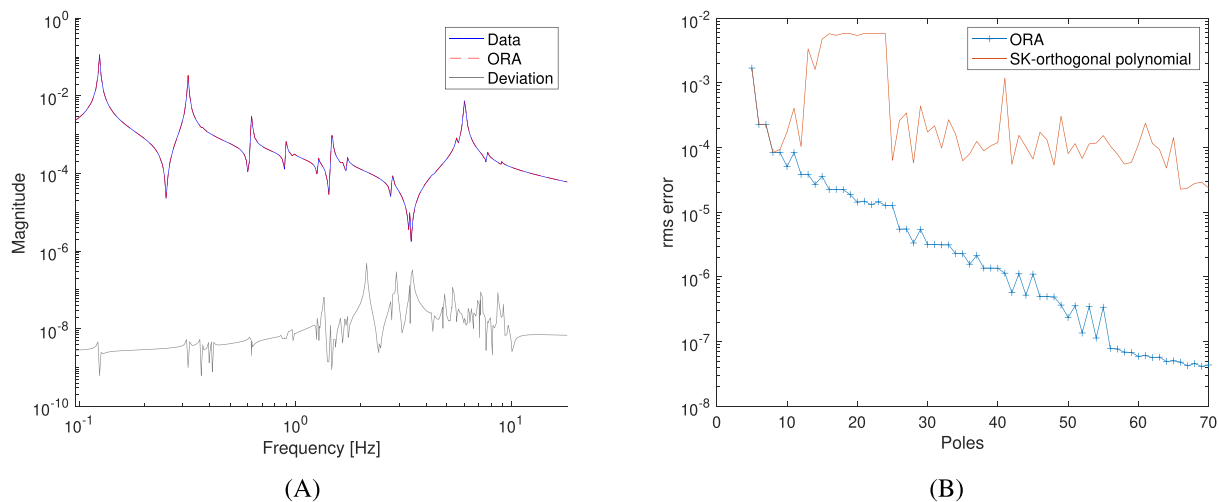
where  $(G_0 G_1 \dots G_n) = (g_1 g_2 \dots g_N)^T$ . After the denominator or the poles are extracted using iteratively calling `denfit`, a final call to `numfit` can provide this state-space model for improper rational functions. Note also how the zeroes of an orthogonal polynomial in (12) are calculated from the eigenvalues of  $A$ .

## 6 | NUMERICAL EXAMPLES

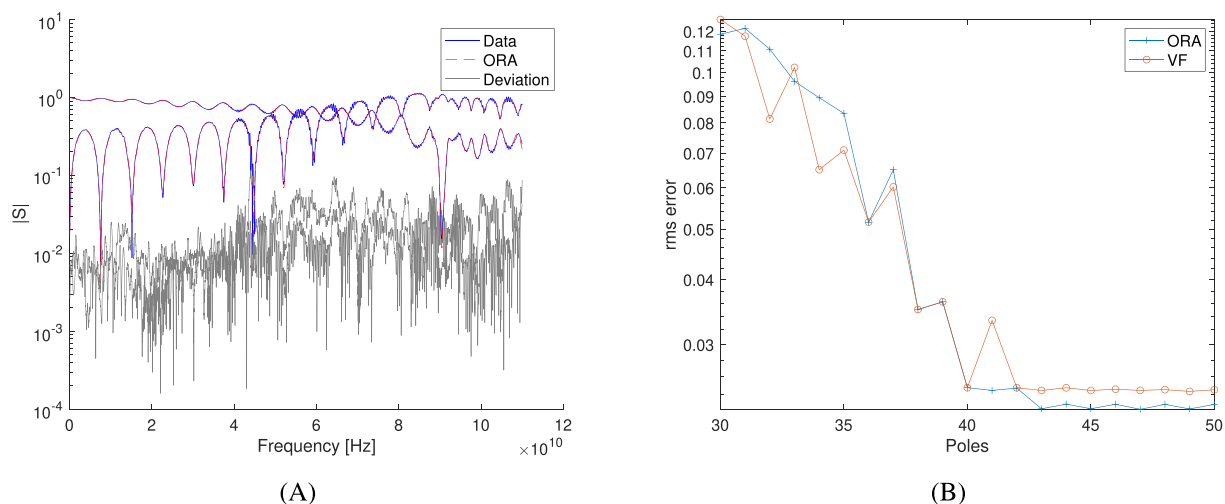
The first example we consider is the ISS 1R module.<sup>2</sup> The data are provided from a state-space model of order 260, so it does not include any noise. Figure 5A shows the original and fitted data for a model using 70 poles. We have assumed that the numerator and denominator polynomials have the same degree for the examples in this paper and have used 20 iterations.

Figure 5B shows a comparison of the rms error as a function of number of poles for the data in Figure 5. We selected the best fit among all the 20 iterations in this figure and started the SK iteration in ORA with  $d(s) = 1$ . SK iteration based on merely an orthogonal polynomial basis becomes severely ill-conditioned due to the division by the denominator<sup>5</sup> as confirmed in Figure 5B. This may be the reason for the poor accuracy observed in earlier implementations of the Arnoldi iteration for rational function approximation<sup>27</sup> and why the method has so far not found widespread popularity. A well-conditioned method is recovered using the orthogonal rational function basis in ORA.

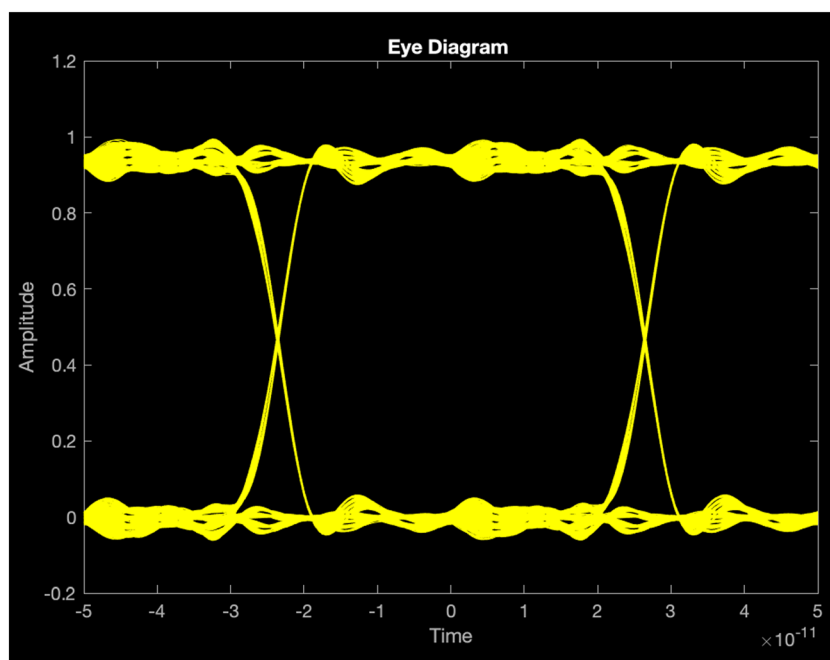
The second example we consider is the noisy data of a stripline measured up to 110 GHz using 5001 frequency points on a vector network analyzer. The first row of the measured 2-port scattering parameters is approximated using 50 poles in ORA as shown in Figure 6A. The proposed ORA method settles down to a lower residual error



**FIGURE 5** (A) Original and fitted data obtained using 70 poles in ORA for the ISS 1R module.<sup>2</sup> The error in magnitude is shown with light gray color. (B) rms error from the best fit among 20 iterations. SK with a merely orthogonal polynomial basis does not provide a well conditioned method [Colour figure can be viewed at [wileyonlinelibrary.com](http://wileyonlinelibrary.com)]



**FIGURE 6** (A) ORA using 50 poles for the first row of the S-parameters of a stripline measured at 5001 frequency points. (B) rms error settles down to a lower level compared to vector fitting [Colour figure can be viewed at [wileyonlinelibrary.com](http://wileyonlinelibrary.com)]

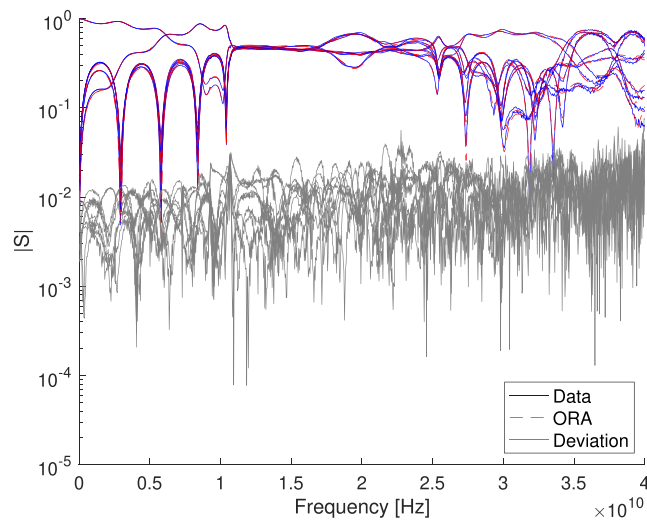


**FIGURE 7** The eye diagram through the measured stripline in Figure 6. A 20-Gbps input signal is used in the simulation. The rational function for time-domain simulation was generated based on the measured transfer function using 34 poles [Colour figure can be viewed at [wileyonlinelibrary.com](http://wileyonlinelibrary.com)]

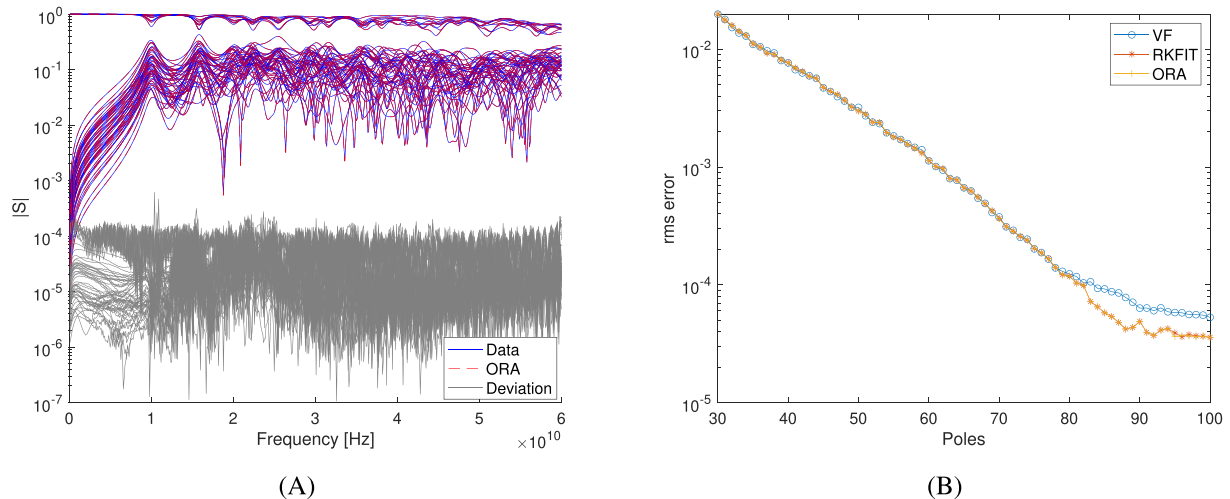
compared to the vector fitting method (vectfit3 implementation on Matlab) as shown in Figure 6B as the number of poles increases.

As an example application of ORA for high-speed circuits, the eye diagram through this stripline is simulated. One option for such transient analysis is exporting a circuit model synthesized from the rational function.<sup>50</sup> This allows transient simulation of high-speed interconnects in the presence of nonlinear circuits. For this example, the eye diagram was generated based on time-domain response of the rational function using Matlab's `timeresp` function. A 20-Gbps input signal is used in the simulation as shown in Figure 7. The rational function was generated based on the measured transfer function using 34 poles.

The third example is a common-mode filter measured up to 40 GHz using 1001 frequency points. The upper triangular portion of the measured 4-port scattering matrix is approximated. Figure 8 shows good fit with a relatively flat residual using 40 poles.



**FIGURE 8** Approximation of the upper triangular portion of the 4-port scattering matrix of a common-mode filter measured at 1001 frequency points. Good fit is obtained using 40 common poles [Colour figure can be viewed at [wileyonlinelibrary.com](https://onlinelibrary.wiley.com/doi/10.1002/ca.3488)]



**FIGURE 9** (A) ORA using 100 poles for the upper triangular portion of the S-parameters of a 10-port cavity resonator simulated at 600 frequency points. (B) ORA was able to find solutions with lower rms error compared to vector fitting as the model order increases [Colour figure can be viewed at [wileyonlinelibrary.com](https://onlinelibrary.wiley.com/doi/10.1002/ca.3488)]

The final example is a cavity resonator simulated at ten ports using a full-wave simulator. The upper triangular portion of the scattering parameters (55 elements total) are approximated using ORA as shown in Figure 9A. For this example, the proposed ORA method was also able to find solutions with lower residual error compared to the vector fitting method. ORA and RKFIT provided similar accuracy. A complete convergence analysis is not yet available for ORA, vector fitting, or RKFIT.<sup>39</sup> The residual error does not monotonously decrease with the number iterations or the number of poles. Therefore, in Figure 9B, the best approximations among all 20 iterations were chosen for each model order. The total run time for Figure 9B on a laptop with Intel i7 processor was fastest with 151s for ORA, 282s for RKFIT, and 325s for vector fitting.

## 7 | CONCLUSIONS

This paper introduced the ORA method for rational function approximation. The method is an extension of the recently developed Vandermonde with Arnoldi and stabilized SK methods to ensure real polynomial coefficients and stable

poles for realizability of the rational functions. The new method is also presented for multiport networks and applied on rational function approximations of measured or simulated scattering parameters. For the considered examples, ORA showed a trend to find solutions with better accuracy compared to vector fitting as the model order is increased, where a 10-port model approximation also showed a speed up of approximately two times. The presented method does not require an initial selection of poles and is well-conditioned due to the orthogonalization of rational functions in the SK iteration.

## DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

## ORCID

Arif Ege Engin  <https://orcid.org/0000-0002-1598-4706>

## REFERENCES

1. Gustavsen B, Semlyen A. Rational approximation of frequency domain responses by vector fitting. *IEEE Trans Power Deliv.* 1999;14(3):1052-1061.
2. Gugercin S, Antoulas AC, Bedrossian N. Approximation of the international space station 1R and 12A models. In: Proceedings of the 40th IEEE Conference on Decision and Control (Cat. No. 01ch37228), Vol. 2; 2001:1515-1516.
3. Sarkar TK, Salazar-Palma M, Mokole EL. Application of the principle of analytic continuation to interpolate/extrapolate system responses resulting in reduced computations—part A: parametric methods. *IEEE J Multiscale Multiphysics Comput Tech.* 2016;1:48-59. doi:10.1109/JMMCT.2016.2613868
4. Austin AP, Krishnamoorthy M, Leyffer S, Mrenna S, Müller J, Schulz H. Practical algorithms for multivariate rational approximation. *Comput Phys Commun.* 2021;261:107663. doi:10.1016/j.cpc.2020.107663
5. Hokanson JM. Multivariate rational approximation using a stabilized Sanathanan-Koerner iteration; 2020.
6. Swaminathan R. The next frontier: enabling Moore's law using heterogeneous integration. *Chip Scale Rev.* 2022;26(3):11-22.
7. Khan MSM, Xi C, Khan AA, Rahman MT, Tehranipoor MM, Asadizanjani N. Secure interposer-based heterogeneous integration. *IEEE Design and Test.* 2022;39(6):1-1. doi:10.1109/MDAT.2022.3151020
8. Sheikh F, Nagisetty R, Karnik T, Kehlet D. 2.5D and 3D heterogeneous integration: emerging applications. *IEEE Solid-State Circ Mag.* 2021;13(4):77-87. doi:10.1109/MSSC.2021.3111386
9. Samanta KK. Pushing the envelope for heterogeneity: Multilayer and 3-D heterogeneous integrations for next generation millimeter- and submillimeter-wave circuits and systems. *IEEE Microwave Magazine.* 2017;18(2):28-43. doi:10.1109/MMM.2016.2635858
10. Gupta P, Iyer SS. Goodbye, motherboard. bare chiplets bonded to silicon will make computers smaller and more powerful: hello, silicon-interconnect fabric. *IEEE Spectrum.* 2019;56(10):28-33. doi:10.1109/MSPEC.2019.8847587
11. Engin AE, Mathis W, John W, Sommer G, Reichl H. Closed-form network representations of frequency-dependent RLGC parameters. *Int J Circ Theory Appl.* 2005;33:463-485.
12. Engin AE, Ndip I, Lang K, Aguirre J. Closed-form multipole debye model for time-domain modeling of lossy dielectrics. *IEEE Trans Electromagn Compat.* 2019;61(3):966-968. doi:10.1109/TEMC.2018.2838522
13. Chou C-C, Schutt-Ainé JE. Equivalent circuit synthesis of multiport S parameters in pole-residue form. *IEEE Trans Components Packag Manuf Technol.* 2021;11(11):1971-1979. doi:10.1109/TCPMT.2021.3115113
14. Triverio P. 8 vector fitting. *8 Vector fitting: De Gruyter;* 2021:275-310.
15. Nouri B, Gad E, Nakhla M, Achar R. *4 Model Order Reduction in Microelectronics: De Gruyter;* 2020:111-144.
16. Ioan D, Ciuprina G, Schilders WHA. *5 Complexity Reduction of Electromagnetic Systems: De Gruyter;* 2020:145-200.
17. Mayo AJ, Antoulas AC. A framework for the solution of the generalized realization problem. *Linear Algebra Appl.* 2007;425(2):634-662. Special Issue in honor of Paul Fuhrmann. doi:10.1016/j.laa.2007.03.008
18. Sanathanan C, Koerner J. Transfer function synthesis as a ratio of two complex polynomials. *IEEE Trans Autom Control.* 1963;8(1):56-58. doi:10.1109/TAC.1963.1105517
19. Nakatsukasa Y, Sète O, Trefethen LN. The AAA algorithm for rational approximation. *SIAM J Sci Comput.* 2018;40(3):A1494-A1522. doi:10.1137/16M1106122
20. Gosea IV, Antoulas AC. Rational approximation of the absolute value function from measurements: a numerical study of recent methods; 2020.
21. Valera-Rivera A, Engin AE. AAA algorithm for rational transfer function approximation with stable poles. *IEEE Lett Electromagn Compat Pract Appl.* 2021;3(3):92-95. doi:10.1109/LEMCPA.2021.3104455
22. Gosea IV, Guttel S. Algorithms for the rational approximation of matrix-valued functions. *SIAM J Sci Comput.* 2021;43(5):A3033-A3054.
23. Hendrickx W, Dhaene T. A discussion of "rational approximation of frequency domain responses by vector fitting". *IEEE Trans Power Syst.* 2006;21(1):441-443. doi:10.1109/TPWRS.2005.860905
24. Beyene WT. Improving time-domain measurements with a network analyzer using a robust rational interpolation technique. *IEEE Trans Microw Theory Tech.* 2001;49(3):500-508. doi:10.1109/22.910554

25. Hokanson JM, Magruder CC. Least squares rational approximation. arXiv preprint arXiv:1811.12590; 2018.
26. Brubeck PD, Nakatsukasa Y, Trefethen LN. Vandermonde with Arnoldi. *SIAM Review*. 2021;63(2):405-415. doi:10.1137/19M130100X
27. Deschrijver D, Dhaene T. Univariate rational macromodeling of high speed passive components. *ACES News Lett Paper*. 2005;20(2):35-60.
28. Coelho CP, Phillips JR, Silveira LM. Robust rational function approximation algorithm for model generation. In: Proceedings 1999 Design Automation Conference (Cat. No. 99ch36361); 1999:207-212.
29. Coelho CP, Phillips JR, Silveira LM. Generating high-accuracy simulation models using problem-tailored orthogonal polynomials basis. *IEEE Trans Circ Syst I: Regular Papers*. 2006;53(12):2705-2714. doi:10.1109/TCSI.2006.883865
30. Hochman A, Leviatan Y, White JK. On the use of rational-function fitting methods for the solution of 2D laplace boundary-value problems. *J Comput Phys*. 2013;238:337-358. doi:10.1016/j.jcp.2012.08.015
31. Deschrijver D, Haegeman B, Dhaene T. Orthonormal vector fitting: a robust macromodeling tool for rational approximation of frequency domain responses. *IEEE Trans Adv Packag*. 2007;30(2):216-225. doi:10.1109/TADVP.2006.879429
32. Van gucht P, Bultheel A. Orthogonal rational functions for system identification: numerical aspects. *IEEE Trans Autom Control*. 2003;48(4):705-709. doi:10.1109/TAC.2003.809761
33. Ito S, Nakatsukasa Y. Stable polefinding and rational least-squares fitting via eigenvalues. *Numer Math*. 2018;139(3):633-682. doi:10.1007/s00211-018-0948-4
34. Bultheel A, Van Barel M, Van gucht P. Orthogonal basis functions in discrete least-squares rational approximation. *J Comput Appl Math*. 2004;164-165:175-194. Proceedings of the 10th International Congress on Computational and Applied Mathematics. doi:10.1016/S0377-0427(03)00497-7
35. de Vries DK, Van den Hof PMJ. Frequency domain identification with generalized orthonormal basis functions. *IEEE Trans Autom Control*. 1998;43(5):656-669. doi:10.1109/9.668831
36. Bultheel A, Van gucht P, Van Barel M. Orthogonal rational functions with real coefficients and semiseparable matrices. *J Comput Appl Math*. 2010;233(5):1192-1201. Special Issue Dedicated to William B. Gragg on the Occasion of His 70th Birthday. doi:10.1016/j.cam.2008.11.016
37. Ruhe A. Rational Krylov sequence methods for eigenvalue computation. *Linear Algebra Appl*. 1984;58:391-405. doi:10.1016/0024-3795(84)90221-0
38. Pranić MS, Reichel L, Rodriguez G, Wang Z, Yu X. A rational Arnoldi process with applications. *Numerical Linear Algebra Appl*. 2016;23(6):1007-1022. doi:10.1002/nla.2065
39. Berljafa M, Güttel S. The RKFIT algorithm for nonlinear rational approximation. *SIAM J Sci Comput*. 2017;39(5):A2049-A2071. doi:10.1137/15M1025426
40. Ma A, Deaton D, Engin AE. Stabilized Sanathanan-Koerner iteration for rational transfer function approximation of scattering parameters. IMAPS; 2021.
41. Antoulas AC. Approximation of large-scale dynamical systems: an overview. *IFAC Proc Vol*. 2004;37(11):19-28. 10th IFAC/IFORS/IMACS/IFIP Symposium on Large Scale Systems 2004: Theory and Applications, Osaka, Japan, 26-28 July, 2004 doi: doi:10.1016/S1474-6670(17)31584-7
42. Forsythe GE. Generation and use of orthogonal polynomials for data-fitting with a digital computer. *J Soc Ind Appl Math*. 1957;5(2):74-88.
43. Greif C, Varah JM. Iterative solution of skew-symmetric linear systems. *SIAM J Matrix Anal Appl*. 2009;31:584-601.
44. Barnett S. A companion matrix analogue for orthogonal polynomials. *Linear Algebra Appl*. 1975;12:197-202.
45. Levy EC. Complex-curve fitting. *IRE Trans Autom Control*. 1959;AC-4(1):37-43. doi:10.1109/TAC.1959.6429401
46. Inkilä K. Homogeneous least squares problem. *Photogrammetric J Finl*. 2005;19(2):34-42.
47. Deschrijver D, Mrozowski M, Dhaene T, De Zutter D. Macromodeling of multiport systems using a fast implementation of the vector fitting method. *Microw Wirel Components Lett IEEE*. 2008;18(6):383-385. doi:10.1109/LMWC.2008.922585
48. Bradde T, Grivet-Talocia S, De Stefano M, Zanco A. A scalable reduced-order modeling algorithm for the construction of parameterized interconnect macromodels from scattering responses. In: 2018 IEEE Symposium on Electromagnetic Compatibility, Signal Integrity and Power Integrity (EMC, SI PI); 2018:650-655.
49. Golub GH, van Loan CF. *Matrix Computations*. Fourth: JHU Press; 2013.
50. Swaminathan M, Engin AE. *Power Integrity Modeling and Design for Semiconductors and Systems*. Upper Saddle River, NJ, USA: Prentice Hall Press; 2007.

**How to cite this article:** Ma A, Engin AE. Orthogonal rational approximation of transfer functions for high-frequency circuits. *Int J Circ Theor Appl*. 2022;1-13. doi:10.1002/cta.3488