AAA Algorithm for Rational Transfer Function Approximation With Stable Poles

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Abstract—Signal and power integrity analysis in time-domain requires suitable models for interconnects and power distribution networks, which are often available only in terms of their frequency responses from electromagnetic simulations or measured data. A rational transfer function approximation of such data allows its integration in circuit simulators. Recently the AAA (adaptive Antoulas-Anderson) algorithm has been introduced for rational function approximation. This letter introduces a modification of that algorithm for fitting in terms of the squared variable (s^2), called AAA^2 algorithm, for generating rational transfer functions with stable poles from frequency response available from simulated or measured data.

Index Terms—Macromodeling, stability, causality, vector fitting.

I. INTRODUCTION

RECENTLY the AAA (adaptive Antoulas-Anderson) algorithm has been introduced for rational function approximation [1]. The AAA provides a simple and numerically stable algorithm to obtain a transfer function in barycentric form. The order is obtained through an adaptive approach of increasing the order by one until convergence.

Existing methods for rational transfer function approximation include vector fitting [2], Loewner method [3], and Sanathanan-Koerner (SK) iteration [4]. Vector fitting in particular has been successfully applied in a wide area of applications. Comparative studies have been of great interest as well (see for example AAA vs. the Loewner method in [5]; the barycentric approximation vs. vector fitting in [6]; SK vs. vector fitting in [7]). AAA algorithm offers the advantages that it tends to be better conditioned than a partial-fractions or a monomial representation, and does not require an iterative improvement from an initial selection of poles or nodes.

The AAA algorithm has been primarily applied on approximation of functions available in closed-form. For EMC practice, it can be of use for generating rational transfer functions or equivalent circuit models from simulated or measured data. The generated rational functions can for example be used for signal and power integrity analysis in time domain, where the data represents a passive electrical network. Such rational functions should have real polynomial coefficients and no
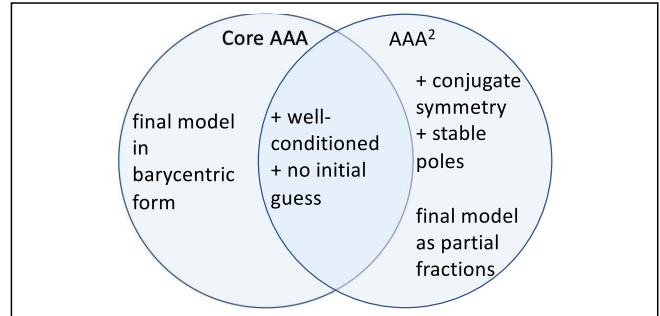
poles with positive real parts, which are however not enforced in the AAA algorithm. This letter introduces a modification of that algorithm for fitting in terms of the squared variable (s^2), called AAA^2 algorithm, for generating rational transfer functions satisfying these properties from frequency response available from simulated or measured data.

The standard approach for generating stable poles in vector fitting is flipping the sign of unstable poles through the pole relocation process. AAA algorithm does not depend on pole relocation, so pole flipping can only be applied at the final step of the algorithm. We also introduce a second method where guaranteed stable poles are obtained without the need for pole flipping.

II. CORE AAA ALGORITHM

The "core" AAA algorithm [1] provides a rational approximation to complex-valued data $H_i = H(j\omega_i)$ in the barycentric form as

$$\frac{n(s)}{d(s)} = \frac{\sum_{i=0}^{N} \frac{a_i H_i}{s - j\omega_i}}{\sum_{i=0}^{N} \frac{a_i}{s - j\omega_i}}. \tag{1}$$

The function $n(s)/d(s)$ interpolates the transfer function at the nodes $s = j\omega_i$ for any non-zero weight a_i. In a practical

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Take-Home Messages:
1) A new method called AAA^2 is introduced to generate rational transfer functions with stable poles from frequency response available from simulated or measured data.
2) The method is well conditioned and requires no guess for the selection of initial nodes or poles.
3) The generated transfer functions can be used for signal and power integrity analysis in time-domain.

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application, there will be additional data points which are used to find the weights a_i that minimize the linearized least squares error of $H(s)d(s) - n(s)$ over those additional data points. The algorithm is based on a greedy approach of increasing order by adding an additional node $j\omega_{N+1}$ where the norm of the residual $H(s) - n(s)/d(s)$ is maximum. The AAA algorithm has also been extended to fitting matrices rather than scalars [8], [9], where the frequency response H_i would represent a matrix.

The linearized least squares solution of the weights in (1) is obtained through a singular value decomposition (SVD) approach. Let $\hat{\omega}$ represent the vector of $M+1$ frequency points separate from the $N+1$ frequency points ω used as nodes in the barycentric form. Similarly, $\hat{H}_k = H(j\hat{\omega}_k)$ are the frequency responses obtained at those additional frequencies. The vector a of weights a_i is obtained by solving the least squares problem

$$\text{minimize} \sum_{k=0}^{M} \left| \sum_{i=0}^{N} \frac{a_i \hat{H}_k}{j\hat{\omega}_k - j\omega_i} - \sum_{i=0}^{N} \frac{a_i H_i}{j\hat{\omega}_k - j\omega_i} \right|, \|a\| = 1 \quad (2)$$

which can be expressed in terms of the $(M+1) \times (N+1)$ Loewner matrix with the elements $A_{ki} = (\hat{H}_k - H_i)/(j\hat{\omega}_k - j\omega_i)$. The solution a is obtained from the final right singular vector of the SVD of A.

It will be useful to define the Loewner matrix A in terms of the Cauchy matrix $C_{ki} = 1/(j\hat{\omega}_k - j\omega_i)$. It can be seen that

$$A = \hat{H}C - CH, \quad (3)$$

where $\hat{H} = \text{diag}(\hat{H}_0, \ldots, \hat{H}_M)$ and $H = \text{diag}(H_0, \ldots, H_N)$.

III. AAA² WITH CONJUGATE SYMMETRY

This letter introduces a variant of AAA algorithm called AAA² with real polynomial coefficients for both $n(s)$ and $d(s)$, and stable poles. Real polynomial coefficients can be enforced by adding complex conjugate versions of each term at both $n(s)$ and $d(s)$ similar to the barycentric representation in [10] as

$$r(s) = \frac{\sum_{i=0}^{N} \frac{a_i H_i}{s - j\omega_i} + \sum_{i=0}^{N} \frac{a_i^* H_i^*}{s + j\omega_i}}{\sum_{i=0}^{N} \frac{a_i}{s - j\omega_i} + \sum_{i=0}^{N} \frac{a_i^*}{s + j\omega_i}}. \quad (4)$$

The weights $a_i = a_i' + ja_i''$ can also be chosen to be real or pure imaginary. We will present that such a choice results in an approximation with significantly different characteristics. Using for example pure imaginary weights $a_i = ja_i''$ we obtain

$$r(s) = \frac{\sum_{i=0}^{N} \frac{a_i \omega_i H_i'}{s^2 + \omega_i^2} + s \sum_{i=0}^{N} \frac{a_i H_i''}{s^2 + \omega_i^2}}{\sum_{i=0}^{N} \frac{a_i \omega_i}{s^2 + \omega_i^2}} \quad (5)$$

where $H_i = H_i' + jH_i''$. The weights a_i, nodes ω_i, and order N are obtained using the greedy approach in the AAA algorithm.

Since the fitting is done in terms of the squared variable (s^2), we call these variants, which ensure conjugate symmetry, the AAA² algorithm.

To obtain the Loewner matrices for AAA², we define a second Cauchy matrix with the conjugate nodes as $C_{ki}^+ = 1/(j\hat{\omega}_k + j\omega_i)$. The Loewner matrix now doubles in width as (A_1, A_2), where $A_1 = \hat{H}C - CH$ is the original Loewner matrix

and $A_2 = \hat{H}C^+ - C^+H^*$ represents the contribution of the complex conjugate nodes. Since we would like to enforce that the weights also come in complex-conjugate pairs, the least squares problem involves the minimization of

$$\|A_1 a + A_2 a^*\| = \|(A_1 + A_2)a' + j(A_1 - A_2)a''\|. \quad (6)$$

The modified Loewner matrix considering complex conjugate nodes can then be expressed as

$$A = ((A_1 + A_2), j(A_1 - A_2)). \quad (7)$$

The real and imaginary parts of the weights a can be obtained from the SVD solution of

$$\text{minimize} \left\| \begin{pmatrix} A' \\ A'' \end{pmatrix} \begin{pmatrix} a' \\ a'' \end{pmatrix} \right\|, \quad \left\| \begin{pmatrix} a' \\ a'' \end{pmatrix} \right\| = 1, \quad (8)$$

where the real and imaginary parts of $A = A' + jA''$ are stacked to ensure that the SVD solution of a' and a'' consist of real elements.

For the example of pure imaginary weights in (5), the real and imaginary parts of the A matrix can be separated as

$$A' = \begin{pmatrix} \frac{\omega_0 \hat{H}_0' - \omega_0 H_0'}{-\hat{\omega}_0^2 + \omega_0^2} & \cdots & \frac{\omega_N \hat{H}_0' - \omega_N H_N'}{-\hat{\omega}_0^2 + \omega_N^2} \\ \vdots & \ddots & \vdots \\ \frac{\omega_0 \hat{H}_M' - \omega_0 H_0'}{-\hat{\omega}_M^2 + \omega_0^2} & \cdots & \frac{\omega_N \hat{H}_M' - \omega_N H_N'}{-\hat{\omega}_M^2 + \omega_N^2} \end{pmatrix} \quad (9)$$

and

$$A'' = \begin{pmatrix} \frac{\omega_0 \hat{H}_0'' - \hat{\omega}_0 H_0''}{-\hat{\omega}_0^2 + \omega_0^2} & \cdots & \frac{\omega_N \hat{H}_0'' - \hat{\omega}_0 H_N''}{-\hat{\omega}_0^2 + \omega_N^2} \\ \vdots & \ddots & \vdots \\ \frac{\omega_0 \hat{H}_M'' - \hat{\omega}_M H_0''}{-\hat{\omega}_M^2 + \omega_0^2} & \cdots & \frac{\omega_N \hat{H}_M'' - \hat{\omega}_M H_N''}{-\hat{\omega}_M^2 + \omega_N^2} \end{pmatrix}. \quad (10)$$

The method can be extended to network matrices, where the A matrix of each element are stacked to obtain the SVD solution of

$$\text{minimize} \left\| \begin{pmatrix} A_0' \\ A_0'' \\ \vdots \\ A_K' \\ A_K'' \end{pmatrix} \begin{pmatrix} a' \\ a'' \end{pmatrix} \right\|, \quad \left\| \begin{pmatrix} a' \\ a'' \end{pmatrix} \right\| = 1 \quad (11)$$

for a network matrix with $K+1$ elements. This results in a transfer function with same poles for all elements.

IV. POLE EXTRACTION

The core AAA algorithm presents a methodology for extraction of poles by solving a generalized eigenvalue problem. We have found that method to be unsatisfactory unless a frequency scaling is applied, which may be due to the presence of constant terms in the matrices. Instead we adapt the method in [11] modified with a similarity transform to yield complex poles in conjugate pairs.

The poles of $r(s)$ in (4) are obtained as the eigenvalues of the matrix $\delta - c^T b$ (after discarding the 0 eigenvalue), where $c =$

$(a'_0, -a''_0, \ldots, a'_N, -a''_N)/\sum_{i=0}^{N} a'_i, \quad b \quad = \quad (0, \omega_0, \ldots, 0, \omega_N),$
and

$$\delta = \begin{pmatrix} 0 & \omega_0 & 0 & \ldots & 0 \\ -\omega_0 & 0 & 0 & \ldots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \ldots & 0 & 0 & \omega_N \\ 0 & \ldots & 0 & -\omega_N & 0 \end{pmatrix} \quad (12)$$

Any unstable poles can be flipped to make them stable as in vector fitting [2].

For the case of pure imaginary weights as in (5), the poles interestingly come in not only complex conjugate pairs but also with both signs. This can be seen by factoring the denominator of $r(s)$ in pole-zero form as

$$\sum_{i=0}^{N} \frac{a_i \omega_i}{s^2 + \omega_i^2} = a \frac{\prod_{i=0}^{N-1} (s^2 - p_i^2)}{\prod_{i=0}^{N} (s^2 + \omega_i^2)} = a \frac{q(s)}{\prod_{i=0}^{N} (s^2 + \omega_i^2)} \quad (13)$$

where p_i are real or come in complex-conjugate pairs. This allows to write $r(s)$ in rational function form

$$r(s) = r_e(s) + r_o(s) = \frac{p_1(s)}{q(s)} + \frac{s p_2(s)}{q(s)} \quad (14)$$

where $p_1(s)$, $p_2(s)$, and $q(s)$ are all even polynomials of degree up to $2N$ with real coefficients. Although it was not our initial intention to separate the real and imaginary parts of $r(j\omega)$, we observe that $r_e(j\omega)$ and $r_o(j\omega)$ are the real and imaginary parts organically separated in the representation of (5). It would seem the zeros of $q(s)$ include both stable and unstable poles of $r(s)$, but the unstable poles actually belong to $r(-s)$ [12] and can be discarded. Unlike [12] our intention is not to fit using real part or magnitude of data only–we make use of both the real and imaginary parts of the data to obtain the poles.

A special case is a pure imaginary pole p, which is simply rejected in rational approximation of scattering parameters in [12]. Impedance or admittance parameters, on the other hand, can have simple poles on the imaginary axis, representing purely reactive portions of the network. In the examples considered in this letter, any pure imaginary poles within the frequency range of provided data are rejected. Any pure imaginary pole pairs outside the frequency range of data are perturbed with the addition of a small negative real part.

A. Residue Extraction and Passivity

We extract the residues by a least squares fitting using the same method as vector fitting [2] to obtain a final model in partial fractions form:

$$r(s) = e + ds + \sum_{i=0}^{N-1} \frac{k_i}{s - p_i}, \quad (15)$$

where k_i is the residue matrix.

After a strictly stable transfer function is obtained, the extracted $r(s)$ needs to be checked for passivity and if necessary modified to ensure passivity. Methods available to enforce passivity include iterative perturbation of residue matrices [13] or Hamiltonian matrices [14], and fitting using positive fractions or linear matrix inequalities [15], [16], [17].
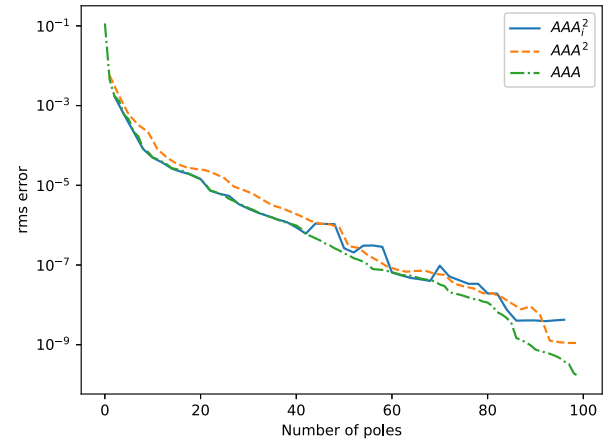
(a)
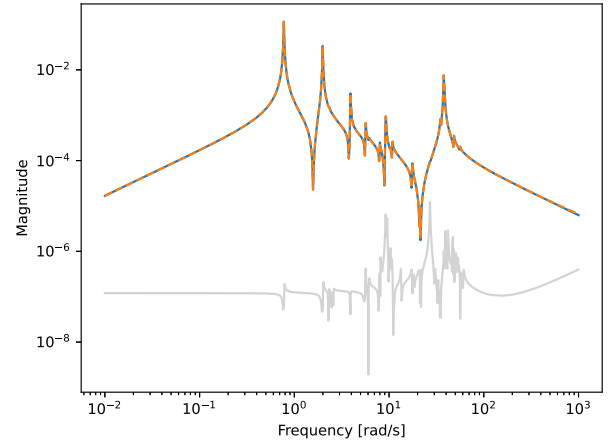
(b)

Fig. 1. (a) rms error for the ISS 1R module; (b) original and fitted data obtained using 40 poles in AAA_i^2, which is based on using pure imaginary weights as in (5). The error in magnitude is shown with light gray color.

V. NUMERICAL EXAMPLES

The first example is the ISS 1R module [18]. The data is provided from a state-space model of order 260, so it does not include any noise. The core AAA and the presented AAA^2 algorithms are used to find a reduced-order model. For AAA, the data was appended with its complex conjugate version. Fig. 1 shows the rms error for models of various orders. The AAA plot in the figure is based on the core AAA algorithm in (1). The AAA^2 stands for the case with general complex weights in (4), whereas AAA_i^2 uses only imaginary weights as in (5). For AAA, the barycentric form was the end model, which does not guarantee stability or conjugate symmetry, whereas AAA^2 and AAA_i^2 error was calculated based on the final partial fractions model (15) with conjugate symmetry and stable poles. Even then the accuracy is comparable to the core AAA algorithm indicating that the new AAA^2 algorithm shares the well conditioning of the core AAA.

Fig. 1 also shows the original and fitted data for a model using 40 poles. The rms error for AAA_i^2 was 8.8×10^{-7}, whereas vector fitting had an rms error of 1.1×10^{-6} after 10 iterations using initial logarithmically spaced poles in complex conjugate pairs as recommended.
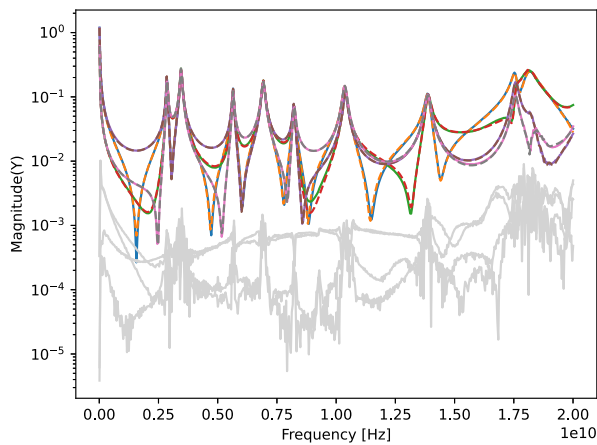
Fig. 2. Magnitude of the elements from the first row of the measured admittance matrix of a common-mode filter with four ports. Rational function (dashed) with 43 poles is compared with the provided data (solid lines). The error in magnitude is shown with light gray color.

The second example is a common-mode filter for differential lines [19]. The measurement is done to obtain admittance parameters of its four ports using a vector network analyzer. Figure 2 shows the comparison of this rational function (dashed) with the provided data (solid lines) for the first row of the admittance parameters. For this case, the AAA^2 algorithm with complex poles is used as AAA_i^2 with imaginary weights resulted in many pure imaginary poles that need to be discarded. The rms error was 0.0013, which was also the rms error from vector fitting after 10 iterations.

VI. Conclusion

The presented AAA^2 algorithm modifies the core AAA algorithm to generate rational transfer functions with real coefficients and stable poles. Numerical examples have demonstrated that the method is able to find rational functions with low order that accurately fit the data. A special case with pure imaginary weights showed good accuracy with noise-free data, whereas AAA^2 with general complex weights can be used for data with or without noise.

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